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On the Existence and Uniqueness of Ground States of a Generalized Spin-Boson Model

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A generalization of the standard spin-boson model is considered. The Hamiltonian $H(\alpha)$ of the model with a coupling parameter $\alpha \in \mathbf{R}$ acts in the tensor product $\mathscr{H} \otimes \mathscr{F}_b$ of a Hilbert space $\mathscr{H}$ and the boson (symmetric) Fock space $\mathscr{F}_b$ over $L^2(\mathbf{R}^\nu)$. The existence and uniqueness of ground states of $H(\alpha)$ are investigated. The degeneracy of the ground states is also discussed. The results obtained are *nonperturbative*. The methods used are those of constructive quantum field theory and the min-max principle. An exact asymptotic formula for the ground state energy of $H(\alpha)$ as $|\alpha| \to \infty$ is also established. © 1997 Academic Press

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1. INTRODUCTION AND MAIN RESULTS

In previous papers [7, 8] we investigated some aspects on the existence and uniqueness of ground states of the standard spin-boson model, which describes a two-level system coupled to a quantized Bose (scalar) field [1, 6, 11, 12, 15, 17, 18, 25, and references therein]. In this paper we propose a generalization of the spin-boson model, which we call a *generalized spin-boson* (*GSB*) *model*, and establish some theorems on the existence and uniqueness of ground states of the GSB model. As the examples given below shows, the GSB model allows us to make a mathematically unified treatment of some specific models of a quantum system linearly coupled to a quantized Bose field. By analysis of the GSB model, one may clarify general mathematical structures underlying those models. In this paper we focus our attention on the existence and uniqueness of ground states of the GSB model. Other aspects of the model (e.g., spectral analysis, resonances, scattering theory) will be discussed elsewhere.

Let

$$\mathscr{F}_b := \mathscr{F}_b(L^2(\mathbf{R}^\nu)) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbf{R}^\nu) \right] \tag{1.1}$$

be the boson (symmetric) Fock space over $L^2(\mathbf{R}^\nu)$ (for a Hilbert space $\mathscr{K}$, $\otimes_s^n \mathscr{K}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathscr{K}$ with convention $\otimes_s^0 \mathscr{K} := \mathbf{C}$, $\nu \in \mathbf{N}$) and $\mathscr{H}$ be a Hilbert space. The Hilbert space for the GSB model is given by

$$\mathscr{F} := \mathscr{H} \otimes \mathscr{F}_b. \tag{1.2}$$

Let $\omega: \mathbf{R}^\nu \to [0, \infty)$ be a Borel measurable function, almost everywhere finite with respect to the Lebesgue measure on $\mathbf{R}^\nu$, physically denoting the dispersion of a free boson in momentum representation,

$$a(f) = \int_{\mathbf{R}^\nu} a(k)\, f(k)^*\, dk, \qquad f \in L^2(\mathbf{R}^\nu), \tag{1.3}$$

be the annihilation operators on $\mathscr{F}_b$ with $a(k)$ the operator-valued distribution kernel of $a(f)$, and

$$H_b := d\Gamma(\omega) = \int_{\mathbf{R}^\nu} \omega(k)\, a(k)^*\, a(k)\, dk, \tag{1.4}$$

the second quantization of the multiplication operator ω on $L^2(\mathbf{R}^\nu)$, denoting the free Hamiltonian of bosons. The Segal field operator

$$\phi(f) := \frac{1}{\sqrt{2}}(a(f) + a(f)^*) \tag{1.5}$$

($f \in L^2(\mathbf{R}^\nu)$) is one of the main objects in the Fock space $\mathscr{F}_b$.

The Hamiltonian of the GSB model we consider is of the form

$$H(\alpha) := A \otimes I + I \otimes H_b + \alpha \overline{\sum_{j=1}^{J} B_j \otimes \phi(\lambda_j)}. \tag{1.6}$$

Here I denotes identity, A is a self-adjoint operator on $\mathscr{H}$, B_j ($j = 1, ..., J$; $J < \infty$) are symmetric operators on $\mathscr{H}$, $\lambda_j \in L^2(\mathbf{R}^\nu)$ ($j = 1, ..., J$), $\alpha \in \mathbf{R}$ is a coupling parameter, and $\bar{T}$ denotes the closure of a closable operator T.

Some examples of the GSB model are given as follows.

Example 1.1 (The Standard Spin-Boson Model). Let $\mu > 0$ be a constant, σ_x, σ_z the standard Pauli matrices and consider the case where $\mathscr{H} = \mathbf{C}^2$, $A = \mu\sigma_z/2$, $J = 1$, $B_1 = \sqrt{2}\,\sigma_x$, $\lambda_1 = \lambda \in L^2(\mathbf{R}^\nu)$. Then $H(\alpha)$ takes the form of the Hamiltonian

$$H_{SB}(\alpha) := \frac{\mu}{2}\sigma_z \otimes I + I \otimes H_b + \sqrt{2}\,\alpha\sigma_x \otimes \phi(\lambda)$$

of the standard spin-boson model, which acts in $\mathbf{C}^2 \otimes \mathscr{F}_b$.

Example 1.2 (An N-Level System Coupled to a Bose Field). Consider the case where $\mathscr{H} = \mathbf{C}^N$ ($N < \infty$) and $J = 1$. Then A and $B := B_1$ can be represented by $N \times N$ Hermitian matrices, so that A has N eigenvalues (counting multiplicity). Hence A describes an unperturbed "atom" with N energy levels. A *positive-temperature version* of this model is discussed in [20]. In that case, however, the Hamiltonian is neither bounded from above nor bounded from below, which is due to that the case of positive temperature is treated. This makes a big difference.

Example 1.3 (A Lattice Spin System Interacting with Phonons). Let Λ be a finite set of the ν-dimensional square lattice $\mathbf{Z}^\nu$ and consider the case where an N component spin $\mathbf{S} = (S^{(1)}, S^{(2)}, ..., S^{(N)})$ sits on each site $i \in \Lambda$ and each component $S^{(n)}$ acts on $\mathbf{C}^s$ with $s \in \mathbf{N}$. The Hilbert space of this spin system is given by $\mathscr{H}_\Lambda = \otimes_{i \in \Lambda} \mathscr{H}_i$ with $\mathscr{H}_i = \mathbf{C}^s$, $i \in \Lambda$. The spin at site i is defined by $\mathbf{S}_i = (S_i^{(1)}, S_i^{(2)}, ..., S_i^{(N)})$, $S_i^{(n)} = I \otimes \cdots \otimes S^{(n)} \otimes \cdots \otimes I$ with

$S^{(n)}$ acting on $\mathscr{H}_i$. A Hamiltonian of the spin system interacting with a Bose field is given by

$$H_\Lambda(\alpha) := \Bigg(-\sum_{(i,j)\subset\Lambda} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\Bigg)\otimes I + I\otimes H_b + \alpha\,\overline{\sum_{j\in\Lambda}\sum_{n=1}^{N} S_j^{(n)}\otimes\phi(\lambda_j^{(n)})}$$

acting in $\mathscr{H}_\Lambda\otimes\mathscr{F}_b$, where $J_{ij}\in\mathbf{R}$, $i, j\in\Lambda$, are constants and $\lambda_j^{(n)}\in L^2(\mathbf{R}^\nu)$, $j\in\Lambda$, $n=1, ..., N$. As is seen, $H_\Lambda(\alpha)$ is of the form (1.6). To our best knowledge, this model has not been discussed in the literature. It would be greatly worth studying as an extension of the usual Heisenberg model.

EXAMPLE 1.4 (Nonrelativistic Particles Interacting with a Bose Field). A Hamiltonian of N nonrelativistic particles with mass $M>0$ in a potential V (a real-valued measurable function on $\mathbf{R}^{\nu N}$) and in interaction with a Bose field is given by

$$H_{PB}(\alpha) = \Bigg(-\frac{1}{2M}\Delta + V\Bigg)\otimes I + I\otimes H_b + \alpha\,\overline{\sum_{j=1}^{J} G_j\otimes\phi(\lambda_j)}$$

acting in $L^2(\mathbf{R}^{\nu N})\otimes\mathscr{F}_b$, where Δ is the Laplacian on $L^2(\mathbf{R}^{\nu N})$ and G_j, $j=1, ..., J$, are symmetric operators on $L^2(\mathbf{R}^{\nu N})$. Related models are discussed in [2–5, 9, 10, 13, 14, 19, 26].

EXAMPLE 1.5 (A Model of a Fermi Field Interacting with a Bose Field). Let $\mathscr{F}_f$ be the fermion Fock space over $L^2(\mathbf{R}^\nu; \mathbf{C}^s)$ $(s\geqslant 1)$, H_f a second quantization operator on $\mathscr{F}_f$ and $\psi(f)$, $f\in L^2(\mathbf{R}^\nu; \mathbf{C}^s)$, the fermion annihilation operators on $\mathscr{F}_f$ (which are bounded). Then a Hamiltonian of a quantum system of a Fermi field interacting with a Bose field is given by

$$H_{FB}(\alpha) = H_f\otimes I + I\otimes H_b + \alpha\,\overline{\sum_{j=1}^{J}\psi(\rho_j)^*\,\psi(\rho_j)\otimes\phi(\lambda_j)}$$

acting in $\mathscr{F}_f\otimes\mathscr{F}_b$, where $\rho_j\in L^2(\mathbf{R}^\nu; \mathbf{C}^s)$, $j=1, ..., J$. In the case $s=2$, this model may serve as a model of electrons interacting with phonons in a metal.

Also various variants of these examples are in the class of the GSB model.

For a linear operator T, we denote by $D(T)$ and $\sigma(T)$ the domain and the spectrum of T respectively. If T is self-adjoint and bounded from below, then we define

$$E_0(T) = \inf\sigma(T). \tag{1.7}$$

For the self-adjointness of $H(\alpha)$, we shall need the following conditions (A.1)–(A.3):

(A.1) The operator A is bounded from below. In this case, we set

$$\mu_0 := E_0(A) \tag{1.8}$$

and

$$\tilde{A} := A - \mu_0, \tag{1.9}$$

which is a nonnegative self-adjoint operator on $\mathscr{H}$.

(A.2) $\lambda_j, \lambda_j/\sqrt{\omega} \in L^2(\mathbf{R}^\nu)$, $j = 1, ..., J$.

(A.3) $D(\tilde{A}^{1/2}) \subset \bigcap_{j=1}^J D(B_j)$ and there exist constants $a_j \geqslant 0$, $b_j \geqslant 0$, $j = 1, ..., J$, such that, for all $u \in D(\tilde{A}^{1/2})$,

$$\|B_j u\| \leqslant a_j \|\tilde{A}^{1/2} u\| + b_j \|u\|, \qquad j = 1, ..., J, \tag{1.10}$$

and

$$|\alpha| \left(\sum_{j=1}^J a_j \left\| \frac{\lambda_j}{\sqrt{\omega}} \right\|_2 \right) < 1, \tag{1.11}$$

where $\|\cdot\|_2$ denotes the norm of $L^2(\mathbf{R}^\nu)$: $\|f\|_2 = \sqrt{\int_{\mathbf{R}^\nu} |f(k)|^2 \, dk}$, $f \in L^2(\mathbf{R}^\nu)$.

Let

$$H_0 = A \otimes I + I \otimes H_b, \qquad H_I = \overline{\sum_{j=1}^J B_j \otimes \phi(\lambda_j)}, \tag{1.12}$$

so that

$$H(\alpha) = H_0 + \alpha H_I. \tag{1.13}$$

By a standard theorem of tensor products of self-adjoint operators (e.g., [21, p. 301, Corollary]), one can easily show that, under condition (A.1), H_0 is self-adjoint and $H_0 \geqslant \mu_0$.

For a vector $v = (v_1, ..., v_J) \in \mathbf{R}^J$ and $f = (f_1, ..., f_J) \in \bigoplus_{j=1}^J L^2(\mathbf{R}^\nu)$, we define

$$M_v(f) = \sum_{j=1}^J v_j \|f_j\|_2. \tag{1.14}$$

We set

$$\lambda = (\lambda_1, ..., \lambda_J) \in \bigoplus_{j=1}^J L^2(\mathbf{R}^\nu) \tag{1.15}$$

and

$$a = (a_1, ..., a_J), \qquad b = (b_1, ..., b_J), \tag{1.16}$$

where a_j and b_j are the constants in (A.3).

For $\theta, \varepsilon, \varepsilon' > 0$, we introduce the following constants:

$$C_{\theta,\varepsilon} = \theta M_a(\lambda/\sqrt{\omega}) + \varepsilon M_a(\lambda), \tag{1.17}$$

$$D_{\theta,\varepsilon'} = \frac{M_a(\lambda/\sqrt{\omega})}{2\theta} + \varepsilon' M_b(\lambda/\sqrt{\omega}), \tag{1.18}$$

$$E_{\varepsilon,\varepsilon'} = \frac{M_a(\lambda)}{8\varepsilon} + \frac{M_b(\lambda/\sqrt{\omega})}{2\varepsilon'} + \frac{M_b(\lambda)}{\sqrt{2}}. \tag{1.19}$$

Let (1.11) be satisfied. If $|\alpha|\ M_a(\lambda/\sqrt{\omega}) \neq 0$, then the interval

$$I_{\alpha,\lambda} := \left(\frac{|\alpha|\ M_a(\lambda/\sqrt{\omega})}{2}, \frac{1}{|\alpha|\ M_a(\lambda/\sqrt{\omega})} \right) \tag{1.20}$$

is not empty. In the case $|\alpha|\ M_a(\lambda/\sqrt{\omega}) = 0$, we set $I_{\alpha,\lambda} := (0, \infty)$. Note that $[1/2, 1] \subset I_{\alpha,\lambda}$ and, for all $\theta \in I_{\alpha,\lambda}$,

$$1 - \theta\ |\alpha|\ M_a(\lambda/\sqrt{\omega}) > 0, \qquad 1 - \frac{|\alpha|\ M_a(\lambda/\sqrt{\omega})}{2\theta} > 0.$$

We define for $\theta \in I_{\alpha,\lambda}$

$$\mathsf{S}_\theta := \{(\varepsilon, \varepsilon') \mid \varepsilon, \varepsilon' > 0, |\alpha|\ C_{\theta,\varepsilon} < 1, |\alpha|\ D_{\theta,\varepsilon'} < 1\} \tag{1.21}$$

and

$$e_\theta(\lambda; \alpha) := \inf_{(\varepsilon,\varepsilon') \in \mathsf{S}_\theta} E_{\varepsilon,\varepsilon'}. \tag{1.22}$$

Then we have

$$e_\theta(\lambda; \alpha) = \frac{|\alpha|\ M_a(\lambda)^2}{8(1 - \theta\ |\alpha|\ M_a(\lambda/\sqrt{\omega}))} + \frac{|\alpha|\ M_b(\lambda/\sqrt{\omega})^2}{2(1 - |\alpha|\ M_a(\lambda/\sqrt{\omega})/2\theta)} + \frac{M_b(\lambda)}{\sqrt{2}} \geqslant 0. \tag{1.23}$$

PROPOSITION 1.1. *Assume* (A.1)–(A.3). *Then the following* (i) *and* (ii) *hold.*

(i) *The Hamiltonian* $H(\alpha)$ *is self-adjoint with* $D(H(\alpha)) = D(H_0) \subset D(H_I)$, *essentially self-adjoint on any core of* H_0 *and bounded from below with*

$$\mu_0 - |\alpha| \inf_{\theta \in I_{\alpha,\lambda}} e_\theta(\lambda; \alpha) \leqslant E_0(H(\alpha)) \leqslant \mu_0. \tag{1.24}$$

(ii) $H(\alpha)$ *is an analytic family of type* (*A*) *near* $\alpha = 0$.

Remark 1.1. Suppose that *each* B_j *is bounded*. Then (A.3) is satisfied with $a_j = 0$, $b_j = \|B_j\|$, $j = 1, ..., J$. Hence

$$e_\theta(\lambda; \alpha) = \frac{|\alpha|}{2} M_b(\lambda/\sqrt{\omega})^2 + \frac{M_b(\lambda)}{\sqrt{2}},$$

so that (1.24) gives

$$\mu_0 - \frac{\alpha^2}{2} M_b(\lambda/\sqrt{\omega})^2 - \frac{|\alpha|\, M_b(\lambda)}{\sqrt{2}} \leqslant E_0(H(\alpha)).$$

In particular,

$$\liminf_{|\alpha| \to \infty} \frac{E_0(H(\alpha))}{\alpha^2} \geqslant -\frac{M_b(\lambda/\sqrt{\omega})^2}{2}. \tag{1.25}$$

For the definition of analytic family of type (A), see [23, p. 16]. Proposition 1.1 will be proved in Section 2. The method of proof is to apply the Kato-Rellich theorem and regular perturbation theory.

In this paper, an eigenvector of a self-adjoint operator T with eigenvalue $E_0(T)$ is called a *ground state* of T (if it exists). We say that T has a (resp. unique) ground state if $\dim \ker(T - E_0(T)) \geqslant 1$ (resp. $\dim \ker(T - E_0(T)) = 1$).

To establish theorems on the existence and uniqueness of ground states of $H(\alpha)$, we need some additional conditions as described below.

(A.4) The function $\omega(k)$ ($k = (k_1, ..., k_\nu) \in \mathbf{R}^\nu$) is continuous with

$$\lim_{|k| \to \infty} \omega(k) = \infty \tag{1.26}$$

and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leqslant C\, |k - k'|^\gamma\, [1 + \omega(k) + \omega(k')], \qquad k, k' \in \mathbf{R}^\nu. \tag{1.27}$$

(A.5) Each λ_j ($j = 1, ..., J$) is continuous.

A typical example of ω satisfying (A.4) is given by $\omega(k)=\sqrt{k^2+m_b^2}$ with a constant $m_b \geqslant 0$ (a case of relativistic bosons; in this case, m_b denotes the rest mass of a boson) or $\omega(k)=k^2/2m_b$ (a case of nonrelativistic bosons).

For a general ω, as the first example above suggests, the constant

$$m := \inf_{k \in \mathbf{R}^\nu} \omega(k) \tag{1.28}$$

may play a role of the "mass" of a boson. In the massive case $m>0$, we can establish a general result on the existence of a ground state of $H(\alpha)$:

THEOREM 1.2. *Let $m>0$. Suppose that* (A.1)–(A.5) *hold and A has compact resolvent. Then $H(\alpha)$ has purely discrete spectrum in $[E_0(H(\alpha)), E_0(H(\alpha))+m)$. In particular, $H(\alpha)$ has a ground state.*

Remark 1.2. If each B_j is bounded, then no restriction on α is needed in Theorem 1.2, since, in this case, (1.11) is automatically satisfied for all $\alpha \in \mathbf{R}$ (see Remark 1.1).

Remark 1.3. If $\dim \mathscr{H} < \infty$, then the conditions (A.1), (A.3) and that A has compact resolvent are trivially satisfied. Hence, in this case, (A.2), (A.4), (A.5) and the condition $m>0$ imply the conclusion of Theorem 1.2. Thus this extends Theorem 1.1 in [7, 8], a corresponding result in the case of the standard spin-boson model.[1]

Remark 1.4. Let $m>0$. Suppose that (A.1)–(A.3) hold and μ_0 is an isolated *simple* eigenvalue of A. Then it follows from Proposition 1.1(ii) and a standard result in regular perturbation theory [23. Theorem XII.9] that, *for all sufficiently small $|\alpha|$, $H(\alpha)$ has a unique ground state and $E_0(H(\alpha))$ is analytic near $\alpha=0$.* Thus, in the massive case $m>0$, the existence and uniqueness of $H(\alpha)$ with $|\alpha|$ sufficiently small can be easily established.

Remark 1.5. For some additional properties of the ground states of $H(\alpha)$ in the case $m>0$, see Section 4.1.

We shall prove Theorem 1.2 in Section 3. The method of the proof is to use a finite volume approximation for $H(\alpha)$ and some limiting arguments.

The massless case $m=0$ is more interesting, but more difficult to analyze than the massive case, because, in the massless case, under condition (A.4), we have $\sigma(H_0)=[\mu_0, \infty)$ (note that $\sigma(\omega)=[0, \infty)$), so that μ_0 cannot be an isolated eigenvalue of H_0 and hence one can not apply the usual regular peturbation theory to establishing the existence of a ground state of $H(\alpha)$ even if $|\alpha|$ is sufficiently small. Also all the other eigenvalues of H_0 (if they

[1] We take this opportunity to make a correction to [7, 8]: in Theorem 1.1 in [7, 8], condition (1.26) also should be supposed.

exist) are embedded in the continuous spectrum of H_0. Thus we are faced with a perturbation problem of embedded eigenvalues. This is a general feature of the spectral problem for a quantum system coupled to a *massless* quantum field. This kind of perturbation problem is difficult to solve in general. As for such a spectral problem, it is only for a few models that the existence of ground states has been established *nonperturbatively* [2–5, 9, 10, 13, 14, 25, 26]. In the present paper, we prove only a partial result on the existence of a ground state of $H(\alpha)$ in the case $m=0$.

Let

$$\mathsf{S}=\{(\theta,\varepsilon,\varepsilon')\mid\theta\in I_{\alpha,\lambda},(\varepsilon,\varepsilon')\in\mathsf{S}_\theta\}\tag{1.29}$$

and, for $(\theta,\varepsilon,\varepsilon')\in\mathsf{S}$,

$$\gamma(\theta,\varepsilon,\varepsilon'):=\frac{E_0(H(\alpha))-\mu_0+|\alpha|\,E_{\varepsilon,\varepsilon'}}{1-|\alpha|\,C_{\theta,\varepsilon}}\tag{1.30}$$

Then, by (1.24),

$$\gamma(\mathbf{s})\geqslant 0,\qquad \mathbf{s}\in\mathsf{S},\tag{1.31}$$

so that we can define

$$\gamma:=\inf_{\mathbf{s}\in\mathsf{S}}\gamma(\mathbf{s})\geqslant 0.\tag{1.32}$$

For a bounded linear operator T, we denote by $\|T\|$ the operator norm of T. Under condition (A.3), $B_j(\tilde{A}+\kappa)^{-1/2}$ is bounded for all $\kappa>0$. Hence we can define

$$\beta_j=\inf_{\kappa>0}\|B_j(\tilde{A}+\kappa)^{-1/2}\|\sqrt{\gamma+\kappa},\qquad j=1,...,J.\tag{1.33}$$

Remark 1.6. It is easy to see that, if each B_j is bounded, then $\beta_j\leqslant\|B_j\|$.

We denote by N_b the boson number operator

$$N_b:=d\Gamma(I)=\int_{\mathbf{R}^\nu}a(k)^*\,a(k)\,dk.\tag{1.34}$$

THEOREM 1.3. *Let* $m=0$. *Suppose that* (A.1)–(A.5) *hold and* A *has compact resolvent. Moreover, suppose that* $\lambda_j/\omega\in L^2(\mathbf{R}^\nu)$, $j=1,...,J$ *and*

$$\alpha^2M_\beta(\lambda/\omega)^2<2.\tag{1.35}$$

where $\beta := (\beta_1, ..., \beta_J)$. *Then* $H(\alpha)$ *has a ground state* Ψ_0 *such that* $\Psi_0 \in D(I \otimes N_b^{1/2})$ *and*

$$\frac{\|I \otimes N_b^{1/2} \Psi_0\|^2}{\|\Psi_0\|^2} \leqslant \frac{\alpha^2 M_\beta(\lambda/\omega)^2}{2 - \alpha^2 M_\beta(\lambda/\omega)^2}. \tag{1.36}$$

Theorem 1.3 will be proved in Section 4. The basic idea of the proof is to obtain a ground state of $H(\alpha)$ as a weak limit of ground states in the massive case (Theorem 1.2). The point of the proof is then to show that the weak limit is not zero. This idea was already developed in our previous papers [7, 8]. The same idea and method are used in [9, 10], which investigate more realistic models of nonrelativistic particles interacting with photons.

To state a result of another type on the existence (and uniqueness) of ground states of $H(\alpha)$, we introduce some objects. The following condition will be needed:

(A.6) The operators A and B_j, $j = 1, ..., J$, are bounded, and B_i and B_j $(i, j = 1, ..., J)$ commute.

This condition implies (A.1) and (A.3). In the rest of this section we assume (A.6).

The operator

$$R_B := \frac{1}{2} \sum_{i,j=1}^{J} \left(\frac{\lambda_i}{\sqrt{\omega}}, \frac{\lambda_j}{\sqrt{\omega}} \right)_2 B_i B_j, \tag{1.37}$$

is a nonnegative bounded self-adjoint operator on $\mathscr{H}$.

In the case $\lambda_j/\omega \in L^2(\mathbf{R}^\nu)$, $j = 1, ..., J$, we define

$$\Lambda := (\Lambda_{ij})_{1 \leqslant i,j \leqslant J}, \qquad \Lambda_{ij} := \left(\frac{\lambda_i}{\omega}, \frac{\lambda_j}{\omega} \right)_2. \tag{1.38}$$

It is easy to see that Λ is a nonnegative Hermitian matrix and that Λ is strictly positive if and only if $\lambda_1, ..., \lambda_J$ are linearly independent. We assume also the following condition.

(A.7) The functions λ_j, $j = 1, ..., J$, are linearly independent with $\lambda_j/\omega \in L^2(\mathbf{R}^\nu)$ and Λ_{ij}, $i, j = 1, ..., J$, are real numbers.

As remarked above, (A.7) implies that Λ is a real, symmetric, strictly positive matrix.

For each $p=(p_1, ..., p_J)\in \mathbf{R}^J$, we define a bounded self-adjoint operator

$$B(p) := \sum_{j=1}^{J} p_j B_j. \tag{1.39}$$

Let

$$L(\alpha) := \frac{1}{\pi^{J/2}\sqrt{\det \Lambda}} \int_{\mathbf{R}^J} e^{-i\alpha B(p)} A e^{i\alpha B(p)} e^{-(p,\, \Lambda^{-1} p)}\, dp - \alpha^2 R_B \tag{1.40}$$

on $\mathscr{H}$. By (A.6) and (A.7), the integral on the right hand side of (1.40) converges in the operator norm topology, defining a bounded self-adjoint operator on $\mathscr{H}$. For a connection of $L(\alpha)$ with $H(\alpha)$, see Remark 5.1 in Section 5. In terms of the operator $L(\alpha)$, we can estimate $E_0(H(\alpha))$:

PROPOSITION 1.4. *Assume* (A.2), (A.6), *and* (A.7). *Then*

$$\mu_0 - \alpha^2 \|R_B\| \leqslant E_0(H(\alpha)) \leqslant E_0(L(\alpha)). \tag{1.41}$$

Moreover,

$$\lim_{|\alpha|\to\infty} \frac{E_0(H(\alpha))}{\alpha^2} = -\|R_B\|. \tag{1.42}$$

Remark 1.7. Under the assumption of Proposition 1.4, we have

$$\|R_B\| \leqslant \frac{M_b(\lambda/\sqrt{\omega})^2}{2}$$

with $b_j = \|B_j\|$, $j=1, ..., J$. Hence (1.41) gives a better lower bound for $E_0(H(\alpha))$ than that of Proposition 1.1 (see Remark 1.1).

Remark 1.8. For all $u\in\mathscr{H}$ with $\|u\|=1$, we have

$$(u, L(\alpha)u) \leqslant \|A\| - \alpha^2 (u, R_B u).$$

Hence, by the variational principle, $E_0(L(\alpha)) \leqslant \|A\| - \alpha^2(u, R_B u)$. Since R_B is a nonnegative bounded self-adjoint operator, we have $\sup_{\|u\|=1,\, u\in\mathscr{H}} (u, R_B u) = \|R_B\|$. Hence

$$E_0(L(\alpha)) \leqslant \|A\| - \alpha^2 \|R_B\|. \tag{1.43}$$

Therefore, if $\|R_B\| \neq 0$ and $\alpha^2 > (\|A\| - \mu_0)/\|R_B\|$, then $E_0(L(\alpha)) < \mu_0$. Thus, at least for such $|\alpha|$, (1.41) gives a more precise upper bound for

$E_0(H(\alpha))$ than that of Proposition 1.1. This fact is more explicitly exemplified in the standard spin-boson model as is shown in the following remark.

Remark 1.9. If A and B_j anticommute, i.e., $AB_j = -B_jA$, $j = 1, ..., J$, then $e^{-i\alpha B(p)}A = Ae^{i\alpha B(p)}$ and hence

$$L(\alpha) = Ae^{-\alpha^2 \sum_{i,j=1}^{J} A_{ij} B_i B_j} - \alpha^2 R_B \tag{1.44}$$

The standard spin-boson model satisfies this assumption (see Example 1.1). Let $L_{SB}(\alpha)$ be the operator $L(\alpha)$ in the case of the standard spin-boson model. Then we have

$$L_B(\alpha) = \frac{\mu}{2}\sigma_z e^{-2\alpha^2 \|\lambda/\omega\|_2^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_2^2, \tag{1.45}$$

which implies that

$$E_0(L_{SB}(\alpha)) = -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega\|_2^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_2^2. \tag{1.46}$$

In the present example we have $\mu_0 = -\mu/2$. Hence, if $(1 - e^{-2\alpha^2 \|\lambda/\omega\|_2^2})\,\mu/2 < \alpha^2 \|\lambda/\sqrt{\omega}\|_2^2$, then $E_0(L_{SB}(\alpha)) < \mu_0 = -\mu/2$.

Remark 1.10. Equation (1.42) is a formula for the asymptotic behavior of the ground state energy $E_0(H(\alpha))$ of $H(\alpha)$ in the strong coupling region $|\alpha| \sim \infty$:

$$E_0(H(\alpha)) \sim -\alpha^2 \|R_B\| \qquad (|\alpha| \to \infty).$$

We shall prove Proposition 1.4 in Section 5.

We define for $\varepsilon > 0$

$$C_\varepsilon(\alpha) = \frac{\alpha^2}{2\varepsilon} \left(\sum_{j=1}^{J} (\|A\|\, \|B_j\| + \|B_jA\|) \left\| \frac{\lambda_j}{\omega^{3/2}} \right\|_2 \right)^2 + \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} (\|A\|\, \|B_j\| + \|B_jA\|) \left\| \frac{\lambda_j}{\omega} \right\|_2, \tag{1.47}$$

where we assume also that

$$\lambda/\omega^{3/2} \in L^2(\mathbf{R}^\nu). \tag{1.48}$$

Let

$$\mu_1 = \inf \sigma(A) \backslash \{\mu_0\} \tag{1.49}$$

and

$$M(\alpha) := \sup_{\varepsilon > 0} K_\varepsilon(\alpha) - \alpha^2 \|R_B\| \tag{1.50}$$

with

$$K_\varepsilon(\alpha) = \min\{m(1-\varepsilon) + \mu_0, \mu_1\} - C_\varepsilon(\alpha), \qquad \varepsilon > 0. \tag{1.51}$$

THEOREM 1.5. *Let* $m > 0$. *Assume* (A.2), (A.6), (A.7) *and* (1.48). *Suppose that* μ_0 *is a simple eigenvalue of* A, $\mu_0 < \mu_1$ *and*

$$E_0(L(\alpha)) < M(\alpha) \tag{1.52}$$

Then $H(\alpha)$ *has a unique ground state.*

Remark 1.11. As is seen in the case of the standard spin-boson model (see (1.46)), (1.52) is generally a *nonperturbative* condition on α. Hence Theorem 1.5 is a nonperturbative result. Under the assumption of Theorem 1.5, condition (1.52) is not empty as is shown below. It is easy to see that

$$\lim_{\alpha \to 0} \|L(\alpha) - A\| = 0, \tag{1.53}$$

which implies

$$\lim_{\alpha \to 0} E_0(L(\alpha)) = E_0(A) = \mu_0. \tag{1.54}$$

Let $0 < \varepsilon < 1$. Then $\delta = \min\{m(1-\varepsilon), \mu_1 - \mu_0\} > 0$. Take $\alpha_0 > 0$ such that $D_0 := \delta - C_\varepsilon(\alpha_0) - \alpha_0^2 \|R_B\| > 0$. Then, for all $|\alpha| \leqslant \alpha_0$, $M(\alpha) \geqslant \mu_0 + D_0$. Hence, for all sufficiently small $|\alpha|$, (1.52) holds.

If $m = 0$, then $K_\varepsilon(\alpha) = \mu_0 - C_\varepsilon(\alpha)$ and hence, by (1.41), (1.52) does not hold if $C_\varepsilon(\alpha) \neq 0$. Thus, for (1.52) to hold in the nontrivial case $C_\varepsilon(\alpha) \neq 0$, the condition $m > 0$ is necessary.

It may happen that $H(\alpha)$ has degenerate ground states. For example, consider the case where $\mathscr{H} = \mathbf{C}^N$ ($N < \infty$), $J = 1$, $\lambda_1 = \lambda$ with $\lambda/\omega \in L^2(\mathbf{R}^\nu)$, and A commutes with $B_1 = B$. Then we can identity as $\mathscr{F} = \bigoplus_{n=1}^N \mathscr{F}_b$. Without loss of generality, we can assume that A and B are diagonal matrices with $A = (\mu_{i-1}\delta_{ij})_{i,j=1,\dots,N}$, $B = (\nu_{i-1}\delta_{ij})_{i,j=1,\dots,N}$, $\mu_0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_{N-1}$, $\nu_0 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_{N-1}$. Hence we have $H(\alpha) = \bigoplus_{n=1}^N H_n(\alpha)$ with $H_n(\alpha) = \mu_{n-1} + H_b + \alpha\nu_{n-1}\phi(\lambda)$, $n = 1, \dots, N$. It is well known that $H_n(\alpha)$ has a unique ground state with ground state energy $E_0(H_n(\alpha)) = \varepsilon_n$, where $\varepsilon_n = \mu_{n-1} - (\nu_{n-1}^2 \alpha^2/2) \|\lambda/\sqrt{\omega}\|_2^2$. Hence, if $E_0 := \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_N$, then $H(\alpha)$ has N ground states with ground state energy E_0. In this case, it is not so difficult to see that (1.52) does not hold.

It is interesting to estimate the degeneracy of the ground states of $H(\alpha)$ in a general case. In this respect, we have the following result:

THEOREM 1.6. *Let* $m>0$ *and* $N:=\dim \mathscr{H}<\infty$. *Assume* (A.2), (A.6), (A.7), *and* (1.48). *Suppose that*

$$E_0(L(\alpha))<m+\mu_0-\alpha^2\,\|R_B\|. \tag{1.55}$$

Then the following hold:

(a) *For all* $\varepsilon\in(0, m+\mu_0-\alpha^2\,\|R_B\|-E_0(L(\alpha)))$, *there are at most* N *eigenvalues* (*counting multiplicity*) *of* $H(\alpha)$ *in the interval* $[E_0(H(\alpha)), m+\mu_0-\alpha^2\,\|R_B\|-\varepsilon)$.

(b) *The Hamiltonian* $H(\alpha)$ *has at most* N *ground states:* $\dim\ker(H(\alpha)-E_0(H(\alpha)))\leqslant N$.

We prove Theorems 1.5 and 1.6 in Sections 6 and 7 respectively. The basic idea for the proofs is to employ the min-max principle.

2. PROOF OF PROPOSITION 1.1

We first prove part (i). Using the well known estimates

$$\|a(f)\psi\|\leqslant\|f/\sqrt{\omega}\|_2\,\|H_b^{1/2}\psi\|, \tag{2.1}$$

$$\|a(f)^*\psi\|\leqslant\|f/\sqrt{\omega}\|_2\,\|H_b^{1/2}\psi\|+\|f\|_2\,\|\psi\|, \tag{2.2}$$

for all $f\in L^2(\mathbf{R}^\nu)$ with $f/\sqrt{\omega}\in L^2(\mathbf{R}^\nu)$ and all $\psi\in D(H_b^{1/2})$, we have

$$\|\phi(f)\psi\|\leqslant\sqrt{2}\,\|f/\sqrt{\omega}\|_2\,\|H_b^{1/2}\psi\|+\frac{1}{\sqrt{2}}\,\|f\|_2\,\|\psi\|,$$
$$\psi\in D(H_b^{1/2}),\qquad f, f/\sqrt{\omega}\in L^2(\mathbf{R}^\nu). \tag{2.3}$$

Let Ω be the Fock vacuum in $\mathscr{F}_b$:

$$\Omega=\{1, 0, 0, ...\} \tag{2.4}$$

and

$$\mathscr{F}_{\text{fin}}(\omega)=\mathscr{L}\{\Omega, a(f_1)^*\cdots a(f_n)^*\,\Omega\mid n\geqslant 1, f_j\in D(\omega), j=1, ..., n\}, \tag{2.5}$$

where $\mathscr{L}\{\cdots\}$ denotes the subspace algebraically spanned by vectors in the set $\{\cdots\}$. The subspace $\mathscr{F}_{\text{fin}}(\omega)$ is dense in $\mathscr{F}_b$ and a core of H_b. We introduce a dense subspace in $\mathscr{F}$

$$\mathscr{D}_\omega = D(A) \,\hat{\otimes}\, \mathscr{F}_{\text{fin}}(\omega), \tag{2.6}$$

where $\hat{\otimes}$ denotes algebraic tensor product. Let $\Psi \in \mathscr{D}_\omega$. Then, by (1.12), (A.3) and (2.3), we can show that

$$\|H_I \Psi\| \leqslant \sqrt{2} M_a(\lambda/\sqrt{\omega})\, \|(I \otimes H_b^{1/2})(\tilde{A}^{1/2} \otimes I)\, \Psi\| + \frac{M_a(\lambda)}{\sqrt{2}}\, \|(\tilde{A}^{1/2} \otimes I)\, \psi\|$$
$$+ \sqrt{2} M_b(\lambda/\sqrt{\omega})\, \|(I \otimes H_b^{1/2})\, \Psi\| + \frac{M_b(\lambda)}{\sqrt{2}}\, \|\Psi\|.$$

Let

$$\tilde{H}_0 = \tilde{A} \otimes I + I \otimes H_b. \tag{2.7}$$

Then it is easy to show that

$$\|(I \otimes H_b^{1/2})(\tilde{A}^{1/2} \otimes I)\, \Psi\| \leqslant \frac{1}{\sqrt{2}}\, \|\tilde{H}_0 \Psi\|. \tag{2.8}$$

We have for all $\varepsilon > 0$

$$\|(\tilde{A}^{1/2} \otimes I)\, \Psi\| \leqslant \|\Psi\|^{1/2}\, \|\tilde{A} \otimes I \Psi\|^{1/2} \leqslant \varepsilon\, \|\tilde{A} \otimes I \Psi\| + \frac{1}{4\varepsilon}\, \|\Psi\|$$
$$\leqslant \varepsilon\, \|\tilde{H}_0 \Psi\| + \frac{1}{4\varepsilon}\, \|\Psi\|.$$

Similarly we have for all $\varepsilon' > 0$

$$\|(I \otimes H_b^{1/2})\, \Psi\| \leqslant \varepsilon'\, \|\tilde{H}_0 \Psi\| + \frac{1}{4\varepsilon'}\, \|\Psi\|.$$

Thus we obtain

$$\|H_I \Psi\| \leqslant F_{\varepsilon, \varepsilon'}(\lambda, \omega)\, \|\tilde{H}_0 \Psi\| + G_{\varepsilon, \varepsilon'}(\lambda, \omega)\, \|\Psi\|, \tag{2.9}$$

where

$$F_{\varepsilon,\varepsilon'}(\lambda,\omega)=M_a(\lambda/\sqrt{\omega})+\frac{\varepsilon}{\sqrt{2}}M_a(\lambda)+\sqrt{2}\varepsilon' M_b(\lambda/\sqrt{\omega}), \tag{2.10}$$

$$G_{\varepsilon,\varepsilon'}(\lambda,\omega)=\frac{M_a(\lambda)}{4\sqrt{2}\varepsilon}+\frac{M_b(\lambda/\sqrt{\omega})}{2\sqrt{2}\varepsilon'}+\frac{M_b(\lambda)}{\sqrt{2}} \tag{2.11}$$

Since $\mathscr{D}_\omega$ is a core of $\tilde{H}_0$, inequality (2.9) extends to all $\Psi\in D(\tilde{H}_0)$, giving at the same time $D(\tilde{H}_0)\subset D(H_I)$. Condition (1.11) implies that $|\alpha|\, F_{\varepsilon,\varepsilon'}(\lambda,\omega)<1$ if ε and ε' are sufficiently small. Thus we can apply the Kato–Rellich theorem [22, p. 162, Theorem X.12] to obtain the first half of Proposition 1.1(i).

To prove (1.24), let $\Psi\in\mathscr{D}_\omega$. Then

$$\begin{aligned}
|(\Psi, H_I\Psi)| &\leqslant \sum_{j=1}^{J} \|B_j\otimes I\Psi\|\;\|I\otimes\phi(\lambda_j)\Psi\| \\
&\leqslant \sum_{j=1}^{J} (a_j\,\|\tilde{A}^{1/2}\otimes I\Psi\|+b_j\,\|\Psi\|) \\
&\quad\times\left(\sqrt{2}\left\|\frac{\lambda_j}{\sqrt{\omega}}\right\|_2 \|I\otimes H_b^{1/2}\Psi\|+\frac{\|\lambda_j\|_2}{\sqrt{2}}\,\|\Psi\|\right) \\
&=\sqrt{2}M_a(\lambda/\sqrt{\omega})\,\|\tilde{A}^{1/2}\otimes I\Psi\|\;\|I\otimes H_b^{1/2}\Psi\| \\
&\quad+\frac{M_a(\lambda)}{\sqrt{2}}\,\|\tilde{A}^{1/2}\otimes I\Psi\|\;\|\Psi\| \\
&\quad+\sqrt{2}M_b(\lambda/\sqrt{\omega})\,\|I\otimes H_b^{1/2}\Psi\|\;\|\Psi\|+\frac{M_b(\lambda)}{\sqrt{2}}\,\|\Psi\|^2.
\end{aligned}$$

We have for all $\theta,\varepsilon,\varepsilon'>0$

$$\sqrt{2}\,\|\tilde{A}^{1/2}\otimes I\Psi\|\;\|I\otimes H_b^{1/2}\Psi\|\leqslant\theta(\Psi,\tilde{A}\otimes I\Psi)+\frac{1}{2\theta}(\Psi, I\otimes H_b\Psi),$$

$$\frac{1}{\sqrt{2}}\,\|\tilde{A}^{1/2}\otimes I\Psi\|\;\|\Psi\|\leqslant\varepsilon(\Psi,\tilde{A}\otimes I\Psi)+\frac{1}{8\varepsilon}\,\|\Psi\|^2,$$

$$\sqrt{2}\,\|I\otimes H_b^{1/2}\Psi\|\;\|\Psi\|\leqslant\varepsilon'(\Psi, I\otimes H_b\Psi)+\frac{1}{2\varepsilon'}\,\|\Psi\|^2.$$

Hence we obtain

$$|(\Psi, H_1\Psi)|\leqslant C_{\theta,\varepsilon}(\Psi,\tilde{A}\otimes I\Psi)+D_{\theta,\varepsilon'}(\Psi, I\otimes H_b\Psi)+E_{\varepsilon,\varepsilon'}\,\|\Psi\|^2, \tag{2.12}$$

where the constants $C_{\theta,\varepsilon}$, $D_{\theta,\varepsilon'}$ and $E_{\varepsilon,\varepsilon'}$ are defined by (1.17)–(1.19). Let $\theta \in I_{\alpha,\lambda}$ and $(\varepsilon, \varepsilon') \in \mathsf{S}_\theta$. Then, by using the inequality

$$(\Psi, H(\alpha)\Psi) \geqslant \mu_0 \|\Psi\|^2 + (\Psi, \tilde{A} \otimes I\Psi) + (\Psi, I \otimes H_b \Psi) - |\alpha|\, |(\Psi, H_I \Psi)|$$

we have

$$(\mu_0 - |\alpha|\, E_{\varepsilon,\varepsilon'})\, \|\Psi\|^2 \leqslant (\Psi, H(\alpha)\,\Psi).$$

By the fact proved above, this inequality extends to all $\Psi \in D(H(\alpha))$. Hence the variational principle gives $E_0(H(\alpha)) \geqslant \mu_0 - |\alpha|\, E_{\varepsilon,\varepsilon'}$. Thus the first inequality on (1.24) follows.

By the variational principle and the fact that

$$H_b \Omega = 0, \qquad a(f)\Omega = 0, \qquad f \in L^2(\mathbf{R}^\nu), \tag{2.13}$$

we have for all $u \in D(A)$ with $\|u\| = 1$

$$E_0(H(\alpha)) \leqslant (u \otimes \Omega, H(\alpha) u \otimes \Omega) = (u, Au).$$

Hence $E_0(H(\alpha)) \leqslant \inf_{u \in D(A),\, \|u\|=1} (u, Au) = E_0(A) = \mu_0$. Thus the second inequality on (1.24) follows.

Part (ii) follows from (2.9) and a general fact [23, p. 16, Lemma]. ∎

3. PROOF OF THEOREM 1.2

The basic idea of the proof is to apply methods used in the constructive quantum field theory (e.g., [16]). Throughout this section, the assumption of Theorem 1.2 is taken for granted.

3.1. *A Finite Volume Approximation*

Let $V > 0$ be a parameter and

$$\Gamma_V = \frac{2\pi \mathbf{Z}^\nu}{V} = \left\{ k = (k_1, ..., k_\nu) \,\middle|\, k_j = \frac{2\pi n_j}{V}, n_j \in \mathbf{Z}, j = 1, ..., \nu \right\}. \tag{3.1}$$

Let

$$\mathscr{F}_{b,V} = \mathscr{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n l^2(\Gamma_V) \right], \tag{3.2}$$

the boson Fock space over $l^2(\Gamma_V)$, which describes state vectors of bosons in the finite box $[-V/2, V/2]^\nu$. Each element $\Psi^{(n)}$ in $\bigotimes_s^n l^2(\Gamma_V)$ can be identified with a piecewise constant function in $\bigotimes_s^n L^2(\mathbf{R}^\nu)$ which is a constant on each cube of volume $(2\pi/V)^{n\nu}$ centered about a lattice point $k=(k_1, ..., k_n) \in \Gamma_V^n$. With this identification, $\mathscr{F}_{b,V}$ is regarded as a closed subspace of $\mathscr{F}_b$.

Let

$$C(k, V)=\left[k_1-\frac{\pi}{V}, k_1+\frac{\pi}{V}\right] \times \cdots \times \left[k_\nu-\frac{\pi}{V}, k_\nu+\frac{\pi}{V}\right], \qquad k \in \Gamma_V, \quad (3.3)$$

and

$$\chi_{k,V} := \chi_{C(k,V)},$$

the characteristic function of $C(k, V)$. For each $k \in \Gamma_V$, we introduce

$$a_V(k)=\left(\frac{V}{2\pi}\right)^{\nu/2} a(\chi_{C(k,V)})=\left(\frac{V}{2\pi}\right)^{\nu/2} \int_{[-\pi/V, \pi/V]^\nu} a(k+l)\, dl. \quad (3.4)$$

It is easy to see that, for all $k, l \in \Gamma_V$,

$$[a_V(k), a_V(l)^*]=\delta_{kl}, \qquad [a_V(k), a_V(l)]=0, \qquad [a_V(k)^*, a_V(l)^*]=0 \quad (3.5)$$

on the finite particle subspace

$$\mathscr{F}_0 := \{\psi=\{\psi^{(n)}\}_{n=0}^{\infty} \in \mathscr{F}_b \mid \psi^{(n)}=0 \text{ for all but finitely many } n\}, \quad (3.6)$$

where $[\cdot, \cdot]$ denotes the commutator: $[S, T] := ST-TS$.

We define

$$\omega_V(k)=\omega(k_V), \qquad k \in \mathbf{R}^\nu, \quad (3.7)$$

with k_V a lattice point closed to k:

$$k_V \in \Gamma_V, \qquad |k_j-(k_V)_j| \leqslant \frac{\pi}{V}, \qquad j=1, ..., \nu. \quad (3.8)$$

Let

$$H_{b,V}=d\Gamma(\omega_V)=\int d^\nu k \omega_V(k)\, a(k)^*\, a(k). \quad (3.9)$$

Let C, γ be the constants in (A.4) and set

$$C_V=C\nu^{\gamma/2}\left(\frac{\pi}{V}\right)^\gamma \left(\frac{1}{2m}+1\right). \quad (3.10)$$

In what follows we assume that

$$C_V < 1. \tag{3.11}$$

This is satisfied for all sufficiently large V.

LEMMA 3.1. *We have*

$$D(H_{b,V}) = D(H_b) \tag{3.12}$$

and

$$\|(H_b - H_{b,V})\Psi\| \leqslant \frac{2C_V}{1-C_V} \|H_b\Psi\|, \qquad \Psi \in D(H_b). \tag{3.13}$$

Proof. By (A.4) and (3.8) we have

$$|\omega(k) - \omega_V(k)| \leqslant C\nu^{\gamma/2}(\pi/V)^\gamma \, [1 + \omega(k) + \omega_V(k)].$$

Since

$$1 \leqslant \frac{\omega(k) + \omega_V(k)}{2m}, \qquad k \in \mathbf{R}^\nu,$$

we obtain

$$|\omega(k) - \omega_V(k)| \leqslant C_V[\omega(k) + \omega_V(k)], \qquad k \in \mathbf{R}^\nu. \tag{3.14}$$

It follows from this inequality that

$$(1 - C_V)\,\omega(k) \leqslant (1 + C_V)\,\omega_V(k), \quad (1 - C_V)\,\omega_V(k) \leqslant (1 + C_V)\,\omega(k), \tag{3.15}$$

which imply (3.12). The right hand side of (3.14) is less than or equal to $2C_V\omega(k) + C_V\,|\omega_V(k) - \omega(k)|$. Hence

$$|\omega(k) - \omega_V(k)| \leqslant \frac{2C_V}{1-C_V}\,\omega(k),$$

which implies (3.13). ∎

For $K > 0$, we define functions $\lambda_{j,K,V}$ and $\lambda_{j,K}$ on $\mathbf{R}^\nu$ ($j = 1, ..., J$) by

$$\lambda_{j,K,V}(k) = \sum_{l \in \Gamma_V,\, |l_i| \leqslant K,\, i=1,\, ...,\, \nu} \lambda_j(l)\,\chi_{l,V}(k), \tag{3.16}$$

$$\lambda_{j,K}(k) = \chi_K(k_1) \cdots \chi_K(k_\nu)\,\lambda_j(k), \tag{3.17}$$

where χ_K is the characteristic function of the interval $[-K, K]$.

LEMMA 3.2. *For all* $j=1, ..., J$,

$$\lim_{V\to\infty} \|\lambda_{j,K,V}-\lambda_{j,K}\|_2=0, \tag{3.18}$$

$$\lim_{K\to\infty} \|\lambda_{j,K}-\lambda_j\|_2=0, \tag{3.19}$$

$$\lim_{V\to\infty} \left\|\frac{\lambda_{j,K,V}}{\sqrt{\omega_V}}-\frac{\lambda_{j,K}}{\sqrt{\omega}}\right\|_2=0, \tag{3.20}$$

$$\lim_{K\to\infty} \left\|\frac{\lambda_{j,K}}{\sqrt{\omega}}-\frac{\lambda_j}{\sqrt{\omega}}\right\|_2=0, \tag{3.21}$$

Proof. Using the continuity of λ_j, we can show that, for each $k\in\mathbf{R}^\nu$, $\lambda_{j,K,V}(k)\to\lambda_{j,K}(k)$ as $V\to\infty$. Let $V>V_0>0$. Then

$$C(K):=\max_{j=1,...,J}\sum_{|k_i|\leqslant K+(\pi/V_0),\, i=1,...,\nu}|\lambda_j(k)|<\infty,$$

since λ_j is continuous on $\mathbf{R}^\nu$. Hence

$$|\lambda_{j,K,V}(k)|\leqslant C(K)\,\chi_{[-K-(\pi/V_0),\,K+(\pi/V_0)]^\nu}(k),$$
$$|\lambda_{j,K}(k)|\leqslant C(K)\,\chi_{[-K,K]^\nu}(k), \qquad k\in\mathbf{R}^\nu.$$

By these facts, we can apply the Lebesgue dominated convergence theorem to obtain (3.18). Similarly we can prove (3.19). In the present case, we have $\omega(k), \omega_V(k)\geqslant m$, $k\in\mathbf{R}^\nu$. Hence (3.20) and (3.21) follow from (3.18) and (3.19) respectively. ∎

We define

$$H_{K,V}=H_{0,V}+\alpha H_{I,K,V}, \tag{3.22}$$

$$H_K=H_0+\alpha H_{I,K}, \tag{3.23}$$

where

$$H_{0,V}=A\otimes I+I\otimes H_{b,V}, \tag{3.24}$$

$$H_{I,K,V}=\overline{\sum_{j=1}^{J} B_j\otimes\phi(\lambda_{j,K,V})}, \tag{3.25}$$

$$H_{I,K}=\overline{\sum_{j=1}^{J} B_j\otimes\phi(\lambda_{j,K})}, \tag{3.26}$$

LEMMA 3.3. (i) H_K *is self-adjoint with* $D(H_K)=D(H_0)\subset D(H_{I,K})$, *bounded from below and essentially self-adjoint on any core of* H_0.

(ii) *For all sufficiently large V, $H_{K,V}$ is self-adjoint with $D(H_{K,V}) = D(H_0) \subset D(H_{I,K,V})$, bounded from below, and essentially self-adjoint on any core of $H_{0,V}$.*

Proof. (i) This is just a special case of Proposition 1.1(i), i.e., Proposition 1.1 with λ_j replaced by $\lambda_{j,K}$ (note that (1.11) implies that $|\alpha| (\sum_{j=1}^{J} a_j \|\lambda_{j,K}/\sqrt{\omega}\|_2) < 1$ for all $K > 0$).

(ii) By the proof of Proposition 1.1 (see (2.9)), we have for all $\varepsilon, \varepsilon' > 0$

$$\|H_{I,K,V}\Psi\| \leqslant F_{\varepsilon,\varepsilon'}(\lambda_V^{(K)}, \omega_V) \|\tilde{H}_{0,V}\Psi\| + G_{\varepsilon,\varepsilon'}(\lambda_V^{(K)}, \omega_V) \|\Psi\|, \quad \Psi \in D(H_{0,V}),$$

where $\tilde{H}_{0,V} = \tilde{A} \otimes I + I \otimes H_{b,V}$ and

$$\lambda_V^{(K)} = (\lambda_{1,K,V}, ..., \lambda_{J,K,V}).$$

By (3.18) and (3.20), we have

$$\lim_{V\to\infty} F_{\varepsilon,\varepsilon'}(\lambda_V^{(K)}, \omega_V) = F_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega), \qquad \lim_{V\to\infty} G_{\varepsilon,\varepsilon'}(\lambda_V^{(K)}, \omega_V) = G_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega),$$

where $\lambda^{(K)} := (\lambda_{1,K}, ..., \lambda_{J,K})$. Hence, by condition (1.11), for all sufficiently large V and sufficiently small $\varepsilon, \varepsilon' > 0$, we have $|\alpha| F_{\varepsilon,\varepsilon'}(\lambda_V^{(K)}, \omega_V) < 1$. Hence, by applying the Kato–Rellich theorem and Lemma 3.1, we obtrain the desired result. ∎

LEMMA 3.4. *There exist constants $c > 0$ and $d > 0$ independent of $K \leqslant \infty$ such that*

$$\|\tilde{H}_0\Psi\| \leqslant c\, \|H_K\Psi\| + d\, \|\Psi\|, \qquad \Psi \in D(H_0),$$

where $H_\infty := H(\alpha)$. In particular, for all $z \in \mathbf{C}\backslash\mathbf{R}$, $\tilde{H}_0(H_K - z)^{-1}$ is bounded.

Proof. By (2.9) with λ_j replaced by $\lambda_{j,K}$, we have for all $\Psi \in D(H_0)$

$$\begin{aligned}\|\tilde{H}_0\Psi\| &= \|H_K\Psi - (\alpha H_{I,K} + \mu_0)\Psi\| \\ &\leqslant \|H_K\Psi\| + |\alpha|\, F_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega)\, \|\tilde{H}_0\Psi\| \\ &\quad + (|\alpha|\, G_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega) + |\mu_0|)\, \|\Psi\|.\end{aligned}$$

Note that

$$F_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega) \leqslant F_{\varepsilon,\varepsilon'}(\lambda, \omega), \qquad G_{\varepsilon,\varepsilon'}(\lambda^{(K)}, \omega) \leqslant G_{\varepsilon,\varepsilon'}(\lambda, \omega).$$

Hence, taking ε and ε' such that $|\alpha|\, F_{\varepsilon,\varepsilon'}(\lambda, \omega) < 1$, we obtain the desired inequality. ∎

LEMMA 3.5. *For all* $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\lim_{V \to \infty} \|(H_{K,V} - z)^{-1} - (H_K - z)^{-1}\| = 0. \tag{3.27}$$

Proof. Let V be sufficiently large so that Lemma 3.3(ii) and (3.11) hold. Then we have

$$(H_{K,V} - z)^{-1} - (H_K - z)^{-1} = L_1(V) + L_2(V),$$

where

$$L_1(V) = (H_{K,V} - z)^{-1} \, (I \otimes H_b - I \otimes H_{b,V})(H_K - z)^{-1},$$
$$L_2(V) = \alpha(H_{K,V} - z)^{-1} \, (H_{I,K} - H_{I,K,V})(H_K - z)^{-1}.$$

By Lemma 3.4 and the easily proven inequality

$$\|I \otimes H_b \Psi\| \leqslant \|\tilde{H}_0 \Psi\|, \qquad \Psi \in D(H_0),$$

$I \otimes H_b(H_K - z)^{-1}$ is bounded. By Lemma 3.1 and the fact that

$$\|(H_{K,V} - z)^{-1}\| \leqslant \frac{1}{|\mathrm{Im}\, z|},$$

we have

$$\|L_1(V)\| \leqslant \frac{2C_V}{|\mathrm{Im}\, z| \, (1 - C_V)} \, \|I \otimes H_b(H_K - z)^{-1}\|.$$

Hence $\lim_{V \to \infty} \|L_1(V)\| = 0$. By (2.9), we have for all $\Psi \in \mathscr{F}$

$$\begin{aligned}
&\|(H_{I,K} - H_{I,K,V})(H_K - z)^{-1} \, \Psi\| \\
&\quad \leqslant F_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) \, \|\tilde{H}_0(H_K - z)^{-1} \, \Psi\| \\
&\quad\quad + G_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) \, \|(H_K - z)^{-1} \, \Psi\|.
\end{aligned}$$

Hence $(H_{I,K} - H_{I,K,V})(H_K - z)^{-1}$ is bounded with

$$\begin{aligned}
&\|(H_{I,K} - H_{I,K,V})(H_K - z)^{-1}\| \\
&\quad \leqslant F_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) \, \|\tilde{H}_0(H_K - z)^{-1}\| \\
&\quad\quad + G_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) \, \|(H_K - z)^{-1}\|.
\end{aligned}$$

By Lemma 3.2, we have

$$\lim_{V \to \infty} F_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) = 0, \qquad \lim_{V \to \infty} G_{\varepsilon,\varepsilon'}(\lambda^{(K)} - \lambda_V^{(K)}, \omega) = 0.$$

Hence $\lim_{V\to\infty} \|(H_{I,K}-H_{I,K,V})(H_K-z)^{-1}\|=0$, which implies that

$$\|L_2(V)\| \leqslant \frac{|\alpha|}{|\mathrm{Im}\, z|} \|(H_{I,K}-H_{I,K,V})(H_K-z)^{-1}\| \to 0 \qquad (V\to\infty).$$

Thus (3.27) follows. ∎

3.2. *Completion of Proof of Theorem* 1.2

The following fact is well known:

LEMMA 3.6. *The operator $H_{b,V}$ is reduced by $\mathscr{F}_{b,V}$ and*

$$H_{b,V} \restriction \mathscr{F}_{b,V} = \sum_{k\in\Gamma_V} \omega(k)\, a_V(k)^*\, a_V(k),$$

the second quantization of $\omega \restriction l^2(\Gamma_V)$ in $\mathscr{F}_{b,V}$.

Let

$$\mathscr{F}_V \equiv \mathscr{H} \otimes \mathscr{F}_{b,V}$$

In what follows we assume that V is sufficiently large so that $H_{K,V}$ is self-adjoint (Lemma 3.3(ii)).

LEMMA 3.7. *The operator $H_{K,V}$ is reduced by $\mathscr{F}_V$.*

Proof. Let P_V be the orthogonal projection onto $\mathscr{F}_V$. We decompose $L^2(\mathbf{R}^\nu)$ as $L^2(\mathbf{R}^\nu)=F_{1V}\oplus F_{1V}^{\perp}$ with $F_{1V}=L^2(\mathbf{R}^\nu)\cap l^2(\Gamma_V)$. For vectors of the form

$$\Phi = u\otimes (a(f_1)^* \cdots a(f_n)^*\, \Omega) \in \mathscr{F}, \qquad u\in\mathscr{H}, \quad f_j\in L^2(\mathbf{R}^\nu), \quad j=1, ..., n,$$

we have

$$P_V\Phi = u\otimes (a(p_V f_1)^* \cdots a(p_V f_n)^*\, \Omega),$$

where p_V is the orthogonal projection from $L^2(\mathbf{R}^\nu)$ onto F_{1V}. By Lemma 3.6, $H_{0,V}$ is reduced by $\mathscr{F}_V$ with the reduced poart given by

$$H_{0,V} \restriction \mathscr{F}_V = A\otimes I + I\otimes (H_{b,V}\restriction \mathscr{F}_{b,V}).$$

Hence, for all $\Psi\in D(H_{0,V})$, $P_V\Psi\in D(H_{0,V})$ and

$$H_{0,V}P_V\Psi = P_V H_{0,V}\Psi.$$

Let $\Psi \in \mathscr{D}_{\omega_V}$. Then we have

$$H_{I, K, V} \Psi = \frac{1}{\sqrt{2}} \sum_{l \in \Gamma_V, |l_i| \leqslant K, i = 1, ..., v} \sum_{j=1}^{J} B_j \otimes \{a(\chi_{l,V})^* \lambda_j(l) + a(\chi_{l,V}) \lambda_j(l)^*\} \Psi.$$

Since $p_V \chi_{l,V} = \chi_{l,V}$, it follows that

$$P_V H_{I,K,V} \Psi = H_{I,K,V} P_V \Psi.$$

Hence $H_{K,V} P_V \Psi = P_V H_{K,V} \Psi$. Since $\mathscr{D}_{\omega_V}$ is core of $H_{K,V}$ (Lemma 3.3(ii)), we can extend this equality to all $\Psi \in D(H_{K,V})$, obtaining $P_V H_{K,V} \subset H_{K,V} P_V$. Thus $H_{K,V}$ is reduced by $\mathscr{F}_V$. ∎

LEMMA 3.8. *Let S be a self-adjoint operator on a Hilbert space $\mathscr{K}$ and bounded from below. Let T be a symmetric operator on $\mathscr{K}$. Suppose that $S+T$ is self-adjoint on $D(S+T) := D(S) \cap D(T)$, bounded from below, and S has compact resolvent. Then $S+T$ has compact resolvent.*

Proof. By the assumption, $S+T$ is closed. Hence it follows from the closed graph theorem that there exists a constant $c > 0$ such that

$$\|S\psi\| + \|T\psi\| \leqslant c \, \|(S+T)\psi\| + c \, \|\psi\|, \qquad \psi \in D(S+T). \tag{3.28}$$

By the compactness of the resolvent of S, the set $F_t(S) := \{\psi \in D(S) \mid \|\psi\| \leqslant 1, \ \|S\psi\| \leqslant t\}$ is compact for all $t \geqslant 0$ [23, Theorem XIII.64]. By the closedness of $S+T$, $F_t(S+T)$ is a closed set. By (3.28), for all $\psi \in F_t(S+T)$, we have $\|S\psi\| \leqslant c(t+1)$, i.e., $\psi \in F_{c(t+1)}(S)$. Hence $F_t(S+T)$ is compact. It follows from [23, Theorem XIII.64] that $S+T$ has compact resolvent. ∎

LEMMA 3.9. *The resuced part $H_{K,V} \upharpoonright \mathscr{F}_V$ has purely discrete spectrum.*

Proof. By (A.4), Lemma 3.6 and [23, Theorem XIII.64], $H_{b,V} \upharpoonright \mathscr{F}_{b,V}$ has compact resolvent. Hence the assumption for A implies that $H_{0,V} \upharpoonright \mathscr{F}_V$ has compact resolvent (apply [23, Theorem XIII.64]). Then, applying Lemma 3.8 to $S = H_{0,V} \upharpoonright \mathscr{F}_V$ and $T = H_{I,K,V} \upharpoonright \mathscr{F}_V$, we see that $H_{K,V} \upharpoonright \mathscr{F}_V$ has compact resolvent. Thus we obtain the desired result. ∎

LEMMA 3.10. *We have*

$$H_{K,V} \upharpoonright \mathscr{F}_V^{\perp} \geqslant E_0(H_{K,V}) + m.$$

Proof. We have $\mathscr{F}_b = \mathscr{F}_{b,V} \otimes \mathscr{F}_b(F_{1V}^{\perp}) = \bigotimes_{j=0}^{\infty} \mathscr{F}^{(j)}$, where $\mathscr{F}^{(j)} = \mathscr{F}_{b,V} \otimes [\bigotimes_s^j F_{1V}^{\perp}]$. Hence $\mathscr{F}_{b,V}^{\perp} = \bigoplus_{j=1}^{\infty} \mathscr{F}^{(j)}$ and $\mathscr{F}_V^{\perp} = \mathscr{H} \otimes \mathscr{F}_{b,V}^{\perp} = \bigoplus_{j=1}^{\infty} \mathscr{H} \otimes \mathscr{F}^{(j)}$. On each $\mathscr{H} \otimes \mathscr{F}^{(j)} = \mathscr{F}_V \otimes [\bigotimes_s^j F_{1V}^{\perp}]$, $H_{K,V}$ has the form $C \otimes I_{\otimes_s^j F_{1V}^{\perp}} + I_{\mathscr{F}_V} \otimes D$ with $C = H_{K,V} \upharpoonright \mathscr{F}_V$ and D is a sum of j copies of

$H_{b,V}$, each acting on a single factor $F_{1V}^{\perp}$ (cf. the proof of Lemma 3.7). Since $D \geqslant jm$ on $\otimes_s^j F_{1V}^{\perp}$, the assertion of the lemma follows. ∎

LEMMA 3.11. *For all $z \in \mathbf{C} \backslash \mathbf{R}$,*

$$\lim_{K \to \infty} \|(H_K - z)^{-1} - (H(\alpha) - z)^{-1}\| = 0.$$

Proof. Similar to the proof of Lemma 3.5. ∎

LEMMA 3.12 [24, Lemma 4.6]. *Let T_n and T be self-adjoint operators on a Hilbert space, which are bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and T_n has purely discrete spectrum in $[E_0(T_n), E_0(T_n) + c)$ with some constant $c > 0$. Then, $\lim_{n \to \infty} E_0(T_n) = E_0(T)$ and T has purely spectrum in $[E_0(T), E_0(T) + c)$.*

We are now ready to prove Theorem 1.2: By Lemmas 3.9 and 3.10, $H_{K,V}$ has purely discrete spectrum in $[E_0(H_{K,V}), E_0(H_{K,V}) + m)$. By this fact and Lemma 3.5, we can apply Lemma 312 to conclude that H_K has purely discrete spectrum in $[E_0(H_K), E_0(H_K) + m)$. By this result and Lemma 3.11, the same argument applies to the pair $\langle H_K, H(\alpha) \rangle$ of self-adjoint operators to conclude that $H(\alpha)$ has purely discrete spectrum in $[E_0(H(\alpha)), E_0(H(\alpha)) + m)$.

4. PROOF OF THEOREM 1.3

To prove Theorem 1.3, we first need to investigate some properties of the ground states of $H(\alpha)$ with $m > 0$, which are also physically interesting. Throughout this section, we write simply

$$H := H(\alpha).$$

4.1. *Some Estimates on the Ground States*

LEMMA 4.1. *Assume* (A.1)–(A.3). *Let Ψ_E be an eigenvector of H with eigenvalue E: $H\Psi_E = E\Psi_E$. Let $f \in D(\omega) \cap D(1/\sqrt{\omega})$ and*

$$T(f) = I \otimes a(\omega f) + \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (f, \lambda_j)_2 \, B_j \otimes I.$$

Then $\Psi_E \in D(T(f)) \cap D(I \otimes a(f))$ and $I \otimes a(f)\, \Psi_E \in D(H)$ with

$$(H - E) I \otimes a(f)\, \Psi_E = -T(f)\, \Psi_E.$$

Proof. Since $D(H)=D(H_0)=D(A\otimes I)\cap D(I\otimes H_b)$, $D(A\otimes I)\subset D(B_j\otimes I)$ and $D(I\otimes H_b)\subset D(I\otimes a(\omega f))\cap D(I\otimes a(f))$ (note that $\sqrt{\omega}f$, $f/\sqrt{\omega}\in L^2(\mathbf{R}^\nu)$), it follows that $\Psi_E\in D(T(f))\cap D(I\otimes a(f))$. Let $\Psi\in\mathscr{D}_\omega$. Then we have

$$\begin{aligned}I\otimes a(f)^*\,H\Psi&=[I\otimes a(f)^*, H]\,\Psi+HI\otimes a(f)^*\,\Psi\\&=-T(f)^*\,\Psi+HI\otimes a(f)^*\,\Psi.\end{aligned}$$

Hence

$$\begin{aligned}(I\otimes a(f)\,\Psi_E, H\Psi\Psi)&=-(T(f)\,\Psi_E,\,)+(H\Psi_E, I\otimes a(f)^*\,\Psi)\\&=-(T(f)\,\Psi_E,\,)+E(I\otimes a(f)\,\Psi_E, \Psi).\end{aligned}$$

Since $\mathscr{D}_\omega$ is a core of H (Proposition 1.1(i)), this equality extends to all $\Psi\in D(H)$. Thus the desired result follows. ∎

LEMMA 4.2. *Let $m>0$ and $\{f_l\}_{l=1}^\infty$ be a complete orthonormal system of $L^2(\mathbf{R}^\nu)$ such that $\sqrt{\omega}\,f_l\in L^2(\mathbf{R}^\nu)$, $l\geqslant 1$. Then, for all $\Psi\in D(I\otimes H_b^{1/2})$,*

$$\sum_{l=1}^\infty (I\otimes a(f_l/\sqrt{\omega})\,\Psi, I\otimes a(\sqrt{\omega}\,f_l)\,\Psi)=\|I\otimes N_b^{1/2}\Psi\|^2. \qquad (4.1)$$

Remark 4.1. Let $m>0$. Then we have for all $s>0$

$$D(I\otimes H_b^s)\subset D(I\otimes N_b^s),\quad \|I\otimes N_b^s\Psi\|\leqslant\frac{1}{m^s}\,\|I\otimes H_b^s\Psi\|,\quad \Psi\in D(I\otimes H_b^s). \qquad (4.2)$$

Proof. We can identify $\mathscr{F}$ as $\mathscr{F}=\bigoplus_{n=0}^\infty L^2_{\mathrm{sym}}(\mathbf{R}^{\nu n};\mathscr{H})$, where $L^2_{\mathrm{sym}}(\mathbf{R}^{\nu n};\mathscr{H})$ is the Hilbert space of square integrable $\mathscr{H}$-valued, symmetric functions on $(\mathbf{R}^\nu)^n=\mathbf{R}^{\nu n}$ with convention $L^2_{\mathrm{sym}}(\mathbf{R}^{\nu n};\mathscr{H})\restriction_{n=0}:=\mathscr{H}$. Then one can easily check that, for all $\Psi=\{\Psi^{(n)}\}_{n=0}^\infty\in D(I\otimes a(f))$ $(f\in L^2(\mathbf{R}^\nu))$

$$(I\otimes a(f)\,\Psi)^{(n)}\,(k_1,...,k_n)=\sqrt{n+1}\int_{\mathbf{R}^\nu} f(k)^*\,\Psi^{(n+1)}(k,k_1,...,k_n)\,dk,$$

where the integral on the right hand side is taken in the sense of an $\mathscr{H}$-valued strong Bochner integral. Let $\Psi\in D(I\otimes H_b^{1/2})$. Then we have $I\otimes N_b^{1/2}\Psi=\{\sqrt{n}\,\Psi^{(n)}\}_{n=0}^\infty$ and

$$\sum_{l=1}^{M} (I \otimes a(f_l/\sqrt{\omega})\,\Psi, I \otimes a(\sqrt{\omega}\, f_l)\,\Psi)$$
$$= \sum_{n=0}^{\infty} (n+1) \int \Phi_M^{(n)}(k_1, ..., k_n)\, dk_1 \cdots dk_n,$$

where

$$\Phi_M^{(n)}(k_1, ..., k_n)$$
$$= \sum_{l=1}^{M} \int \frac{\sqrt{\omega(k')}}{\sqrt{\omega(k)}} f_l(k)\, f_l(k')^*$$
$$\times (\Psi^{(n+1)}(k, k_1, ..., k_n), \Psi^{(n+1)}(k', k_1, ..., k_n))_{\mathscr{H}}\, dk\, dk'.$$

Let $\{w_q\}_{q=1}^{N}$ $(N \leqslant \infty)$ be a complete orthonormal system of $\mathscr{H}$ and

$$\Phi_{M,q}^{(n)}(k_1, ..., k_n) = \sum_{l=1}^{M} \left(\int \omega(k)^{-1/2} (\Psi^{(n+1)}(k, k_1, ..., k_n), w_q)_{\mathscr{H}}\, f_l(k)\, dk \right)$$
$$\times \left(\int f_l(k')^* \sqrt{\omega(k')} (w_q, \Psi^{(n+1)}(k', k_1, ..., k_n))_{\mathscr{H}}\, dk' \right).$$

Then we have

$$\Phi_M^{(n)}(k_1, ..., k_n) = \sum_{q=1}^{N} \Phi_{M,q}^{(n)}(k_1, ..., k_n).$$

The interchange between $\int dk\, dk'$ and the sum $\sum_{q=1}^{N}$ can be justified by using

$$\sum_{q=1}^{N} |(\Psi^{(n+1)}(k, k_1, ..., k_n), w_q)_{\mathscr{H}}\, (w_q, \Psi^{(n+1)}(k', k_1, ..., k_n))_{\mathscr{H}}|$$
$$\leqslant \| \Psi^{(n+1)}(k, k_1, ..., k_n)\|_{\mathscr{H}}\, \| \Psi^{(n+1)}(k', k_1, ..., k_n)\|_{\mathscr{H}},$$

which follows from the Schwarz and Bessel inequalities. By the Schwarz inequality on $\sum_{l=1}^{M}$ and the Bessel inequality for the orthonormal system $\{f_l\}_{l=1}^{M}$, we can show that

$$|\Phi_{M,q}^{(n)}(k_1, ..., k_n)| \leqslant \left(\int \omega(k)^{-1} |(\Psi^{(n+1)}(k, k_1, ..., k_n), w_q)_{\mathscr{H}}|^2\, dk \right)^{1/2}$$
$$\times \left(\int \omega(k)\, |(w_q, \Psi^{(n+1)}(k, k_1, ..., k_n))_{\mathscr{H}}|^2\, dk \right)^{1/2}.$$

It follows from the completeness of $\{f_l\}_{l=1}^{\infty}$ that

$$\lim_{M\to\infty} \Phi_{M,q}^{(n)}(k_1, ..., k_n)$$

$$= \int (\Psi^{(n+1)}(k, k_1, ..., k_n), w_q)_{\mathscr{H}} \, (w_q, \Psi^{(n+1)}(k, k_1, ..., k_n))_{\mathscr{H}} \, dk.$$

It is easy to see that

$$\sum_{n=0}^{\infty} \int dk_1 \cdots dk_n (n+1) \sum_{q=1}^{N} \left(\int \omega(k)^{-1} \, |(\Psi^{(n+1)}(k, k_1, ..., k_n), w_q)_{\mathscr{H}}|^2 \, dk \right)^{1/2}$$

$$\times \left(\int \omega(k) \, |(w_q, \Psi^{(n+1)}(k, k_1, ..., k_n))_{\mathscr{H}}|^2 \, dk \right)^{1/2}$$

$$\leqslant \left(\sum_{n=0}^{\infty} \int dk_1 \cdots dk_n \, dk (n+1) \, \omega(k)^{-1} \, \|(\Psi^{(n+1)}(k, k_1, ..., k_n)\|_{\mathscr{H}}^2 \right)^{1/2}$$

$$\times \left(\sum_{n=0}^{\infty} \int dk_1 \cdots dk_n \, dk (n+1) \, \omega(k) \, \|(\Psi^{(n+1)}(k, k_1, ..., k_n)\|_{\mathscr{H}}^2 \right)^{1/2}$$

$$\leqslant \frac{1}{\sqrt{m}} \, \|I \otimes N_b^{1/2} \Psi\| \, \|I \otimes H_b^{1/2} \Psi\| < \infty.$$

Thus, by the dominated convergence theorem, we obtain

$$\lim_{M\to\infty} \sum_{l=1}^{M} (I \otimes a(f_l/\sqrt{\omega}) \, \Psi, I \otimes a(\sqrt{\omega} \, f_l) \, \Psi)$$

$$= \sum_{n=0}^{\infty} (n+1) \int dk \, dk_1 \cdots dk_n$$

$$\times (\Psi^{(n+1)}(k, k_1, ..., k_n), \Psi^{(n+1)}(k, k_1, ..., k_n))_{\mathscr{H}}$$

$$= \|I \otimes N_b^{1/2} \Psi\|^2.$$

Thus (4.1) follows. ∎

LEMMA 4.3. *Let $m > 0$ and assume* (A.1)–(A.3). *Suppose that there exists a ground state Ψ_0 of H. Then $\Psi_0 \in D(I \otimes N_b)$ and*

$$(\Psi_0, I \otimes N_b \Psi_0) \leqslant \frac{\alpha^2}{2} \left(\sum_{j=1}^{J} \left\| \frac{\lambda_j}{\omega} \right\|_2 \|B_j \otimes I \Psi_0\| \right)^2. \tag{4.3}$$

Proof. By (4.2), we have $\Psi_0 \in D(I \otimes N_b)$. Let $f \in D(\omega)$. Then $f/\sqrt{\omega} \in L^2(\mathbf{R}^\nu)$, since $m > 0$. It follows from the fact that $H - E_0(H) \geqslant 0$ and Lemma 4.1 that

$$\begin{aligned} 0 &\leqslant (I \otimes a(f)\, \Psi_0, (H - E_0(H))\, I \otimes a(f)\, \Psi_0) \\ &= -(I \otimes a(f)\, \Psi_0, T(f)\, \Psi_0) \\ &= -(I \otimes a(f)\, \Psi_0, I \otimes a(\omega f)\, \Psi_0) \\ &\quad - \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (f, \lambda_j)_2\, (I \otimes a(f)\, \Psi_0, B_j \otimes I \Psi_0). \end{aligned}$$

Hence

$$\begin{aligned} &(I \otimes a(f)\, \Psi_0, I \otimes a(\omega f)\, \Psi_0) \\ &\quad + \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (f, \lambda_j)_2\, (I \otimes a(f)\, \Psi_0, B_j \otimes I \Psi_0) \leqslant 0. \end{aligned}$$

Let $\{f_l\}_l$ be a complete orthonormal system of $L^2(\mathbf{R}^\nu)$ as in Lemma 4.2. By the inequality just obtained, we have for all $M \in \mathbf{N}$

$$\begin{aligned} &\sum_{l=1}^{M} (I \otimes a(f_l/\sqrt{\omega})\, \Psi_0, I \otimes a(\sqrt{\omega}\, f_l)\, \Psi_0) \\ &\quad + \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (I \otimes a(F_{j,M})\, \Psi_0, B_j \otimes I \Psi_0) \leqslant 0. \end{aligned}$$

where $F_{j,M} = \sum_{n=1}^{M} \omega^{-1/2} (f_l, \lambda_j/\sqrt{\omega})_2\, f_l$. By the well known estimate

$$\|a(f)\psi\| \leqslant \|f\|_2\, \|N_b^{1/2}\psi\|, \qquad f \in L^2(\mathbf{R}^\nu), \quad \psi \in D(N_b^{1/2}), \tag{4.4}$$

Lemma 4.2 and the fact that

$$\lim_{M \to \infty} F_{j,M} = \frac{\lambda_j}{\omega}$$

in $L^2(\mathbf{R}^\nu)$, we obtain

$$(\Psi_0, I \otimes N_b \Psi_0) + \frac{\alpha}{\sqrt{2}} \sum_{j=1}^{J} (I \otimes a(\lambda_j/\omega)\, \Psi_0, B_j \otimes I \Psi_0) \leqslant 0,$$

which implies that $\sum_{j=1}^{J} (I \otimes a(\lambda_j/\omega)\, \Psi_0, B_j \otimes I\Psi_0)$ is real and

$$(\Psi_0, (I \otimes N_b)\, \Psi_0) \leqslant \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} \|I \otimes a(\lambda_j/\omega)\, \Psi_0\|\, \|B_j \otimes I\psi_0\|$$
$$\leqslant \left(\frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} \left\| \frac{\lambda_j}{\omega} \right\|_2 \|B_j \otimes I\Psi_0\| \right) \|I \otimes N_b^{1/2} \Psi_0\|.$$

Hence (4.3) follows. ∎

In what follows, we assume that the assumption of Lemma 4.3 *is satisfied* and Ψ_0 is normalized:

$$\|\Psi_0\| = 1. \tag{4.5}$$

We want to estimate $\|B_j \otimes I\Psi_0\|$.

LEMMA 4.4. *Let γ be defined by* (1.32). *Then*

$$\|\tilde{A}^{1/2} \otimes I\Psi_0\|^2 \leqslant \gamma \tag{4.6}$$

Proof. Using (2.12), we have

$$\begin{aligned}\|\tilde{A}^{1/2} \otimes I\Psi_0\|^2 &= (\Psi_0, \tilde{A} \otimes I\Psi_0)\\ &= (\Psi_0, H\Psi_0) - (\Psi_0, I \otimes H_b \Psi_0) - \alpha(\Psi_0, H_I \Psi_0) - \mu_0\\ &\leqslant E_0(H) - \mu_0 - (\Psi_0, I \otimes H_b \Psi_0) + |\alpha|\, |(\Psi_0, H_I \Psi_0)|\\ &\leqslant E_0(H) - \mu_0 + |\alpha|\, E_{\varepsilon,\varepsilon'} - (1 - |\alpha|\, D_{\theta,\varepsilon'})(\Psi_0, I \otimes H_b \Psi_0)\\ &\quad + |\alpha|\, C_{\theta,\varepsilon}(\Psi_0, \tilde{A} \otimes I\Psi_0).\end{aligned}$$

Hence, for all $(\theta, \varepsilon, \varepsilon') \in \mathsf{S}$,

$$(1 - |\alpha|\, C_{\theta,\varepsilon})\, \|\tilde{A}^{1/2} \otimes I\Psi_0\|^2 \leqslant E_0(H) - \mu_0 + |\alpha|\, E_{\varepsilon,\varepsilon'},$$

from which (4.6) follows. ∎

LEMMA 4.5. *For all $j = 1, ..., J$,*

$$\|B_j \otimes I\Psi_0\| \leqslant \beta_j, \tag{4.7}$$

where β_j, $j = 1, ..., J$, are defined by (1.33).

Proof. For all $\kappa > 0$,

$$\|B_j \otimes I\Psi_0\| \leqslant \|(B_j \otimes I)(\tilde{A} + \kappa)^{-1/2} \otimes I\|\, \|(\tilde{A} + \kappa)^{1/2} \otimes I\Psi_0\|.$$

We have

$$\|(B_j\otimes I)(\tilde{A}+\kappa)^{-1/2}\otimes I\| = \|B_j(\tilde{A}+\kappa)^{-1/2}\|.$$

By Lemma 4.4, we have

$$\|(\tilde{A}+\kappa)^{1/2}\otimes I\Psi_0\|^2 = \|\tilde{A}^{1/2}\otimes I\Psi_0\|^2+\kappa \leqslant \gamma+\kappa.$$

Thus (4.7) follows. ∎

By Lemmas 4.3 and 4.5, we obtain

$$(\Psi_0, (I\otimes N_b)\,\Psi_0) \leqslant \frac{\alpha^2}{2}\, M_\beta(\lambda/\omega)^2. \tag{4.8}$$

Let μ_r, $r=0, 1, 2, ..., q$ $(q\leqslant\infty)$ be distinct eigenvalues of A with $\mu_0<\mu_1<\mu_2<\cdots$ and $\mathscr{H}_r$ be the eigenspace of A with eigenvalue μ_r, so that $\mathscr{H}=\bigoplus_{r=0}^{q}\mathscr{H}_r$ [if $\mathscr{H}$ is finite (resp. infinite) dimensional, then $q<\infty$ (resp. $q=\infty$)]. Let Q_r be the orthogonal projection from $\mathscr{H}$ onto $\bigoplus_{s=0}^{r}\mathscr{H}_s$. We define

$$Q_r^{\perp} = I - Q_r. \tag{4.9}$$

Let P_Ω be the orthogonal projection from $\mathscr{F}_b$ to the one-dimensional subspace $\{z\Omega \mid z\in\mathbf{C}\}$. Our next task is to estimate $(\Psi_0, Q_r\otimes P_\Omega\Psi_0)$.

LEMMA 4.6. *For all* $r=0, ..., q$,

$$Q_r\otimes P_\Omega \geqslant I - I\otimes N_b - Q_r^{\perp}\otimes P_\Omega. \tag{4.10}$$

Proof. We have

$$Q_r\otimes P_\Omega = I\otimes P_\Omega - Q_r^{\perp}\otimes P_\Omega.$$

It is easy to see that

$$P_\Omega \geqslant I - N_b.$$

Hence (4.10) follows. ∎

LEMMA 4.7. *Let*

$$H_I^{(-)} = \frac{1}{\sqrt{2}}\,\overline{\sum_{j=1}^{J} B_j\otimes a(\lambda_j)}. \tag{4.11}$$

Then, for all $r = 0, ..., q-1$,

$$(\Psi_0, Q_r^{\perp} \otimes P_{\Omega} \Psi_0) \leqslant \frac{\alpha^2}{(\mu_{r+1} - E_0(H))^2} \|H_I^{(-)} \Psi_0\|^2 \tag{4.12}$$

Proof. By (2.13), we have

$$Q_r^{\perp} \otimes P_{\Omega} H = Q_r^{\perp} A \otimes P_{\Omega} + \alpha Q_r^{\perp} \otimes P_{\Omega} H_I^{(-)}$$

on $D(H_0)$. By the identity

$$(Q_r^{\perp} \otimes P_{\Omega})(H - E_0(H)) \, \Psi_0 = 0,$$

we obtain

$$(\Psi_0, Q_r^{\perp}(A - E_0(H)) \otimes P_{\Omega} \Psi_0) = -\alpha(Q_r^{\perp} \otimes P_{\Omega} \Psi_0, H_I^{(-)} \Psi_0).$$

By the spectral theorem, we have

$$(\Psi_0, Q_r^{\perp}(A - E_0(H)) \otimes P_{\Omega} \Psi_0) \geqslant (\mu_{r+1} - E_0(H))(\Psi_0, Q_r^{\perp} \otimes P_{\Omega} \Psi_0).$$

Since $\mu_{r+1} > \mu_0 \geqslant E_0(H)$, it follows that

$$(\Psi_0, Q_r^{\perp} \otimes P_{\Omega} \Psi_0) \leqslant \frac{|\alpha|}{\mu_{r+1} - E_0(H)} \|Q_r^{\perp} \otimes P_{\Omega} \Psi_0\| \, \|H_I^{(-)} \Psi_0\|.$$

Thus (4.12) follows. ∎

LEMMA 4.8. *For each* $j = 1, ..., J$, $\Psi_0 \in D(B_j \otimes a(\lambda_j))$ *and*

$$H_I^{(-)} \Psi_0 = \frac{1}{\sqrt{2}} \sum_{j=1}^{J} B_j \otimes a(\lambda_j) \, \Psi_0.$$

Proof. In the same way as in the proof of Proposition 1.1(i), we have for all $\Psi \in \mathscr{D}_{\omega}$ and $j = 1, ..., J$

$$\|B_j \otimes a(\lambda_j) \Psi\| \leqslant \left(\frac{a_j}{\sqrt{2}} + \sqrt{2} \, \varepsilon b_j \right) \|\lambda_j / \sqrt{\omega}\|_2 \, \|\tilde{H}_0 \Psi\|$$

$$+ \frac{b_j}{4\sqrt{2}\,\varepsilon} \|\lambda_j / \sqrt{\omega}\|_2 \, \|\Psi\|. \tag{4.13}$$

Hence

$$\|H_I^{(-)}\Psi\| \leqslant \left(\frac{M_a(\lambda/\sqrt{\omega})}{2} + \varepsilon M_b(\lambda/\sqrt{\omega})\right) \|\tilde{H}_0 \Psi\| + \frac{M_b(\lambda/\sqrt{\omega})}{8\varepsilon} \|\Psi\|, \qquad \Psi \in \mathscr{D}_\omega. \tag{4.14}$$

Since $\mathscr{D}_\omega$ is a core of $\tilde{H}_0$, there exists a sequence $\{\Psi_n\}_{n=1}^\infty \subset \mathscr{D}_\omega$ such that $\Psi_n \to \Psi_0$, $\tilde{H}_0 \Psi_n \to \tilde{H}_0 \Psi_0$ $(n \to \infty)$. Then, by (4.13) and (4.14), $\{B_j \otimes a(\lambda_j)\ \Psi_n\}_n$ and $\{H_I^{(-)}\Psi_n\}_n$ are Cauchy sequences. Since $B_j \otimes a(\lambda_j)$ and $H_I^{(-)}$ are closed, we obtain the desired result. ∎

By Lemma 4.8, we have

$$\|H_I^{(-)}\Psi_0\| \leqslant \frac{1}{\sqrt{2}} \sum_{j=1}^J \|(B_j \otimes I)(\tilde{A}+\kappa)^{-1/2} \otimes I\|\ \|(\tilde{A}+\kappa)^{1/2} \otimes a(\lambda_j)\ \Psi_0\|. \tag{4.15}$$

So we next estimate $\|(\tilde{A}+\kappa)^{1/2} \otimes a(\lambda_j)\ \Psi_0\|$.

By (4.4), we have for all $\Psi \in \mathscr{D}_\omega$

$$\|(\tilde{A}+\kappa)^{1/2} \otimes a(\lambda_j)\,\Psi\|^2 \leqslant \|\lambda_j\|_2^2 \{\kappa\ \|I \otimes N_b^{1/2}\Psi\|^2 + \|(\tilde{A}^{1/2} \otimes I)(I \otimes N_b^{1/2})\,\Psi\|^2\}. \tag{4.16}$$

Moreover, using the fact that

$$[H_b, N_b^{1/2}] = 0$$

on $\mathscr{F}_{\rm fin}(\omega)$, we have

$$\begin{aligned}
&\|(\tilde{A}^{1/2} \otimes I)(I \otimes N_b^{1/2})\,\Psi\|^2 \\
&\quad = (I \otimes N_b^{1/2}\Psi, (\tilde{A} \otimes I)(I \otimes N_b^{1/2})\,\Psi) \\
&\quad = (I \otimes N_b^{1/2}\Psi,\ H - I \otimes H_b - \alpha H_I - \mu_0)(I \otimes N_b^{1/2})\,\Psi) \\
&\quad = \alpha(I \otimes N_b^{1/2}\Psi, [H_I, I \otimes N_b^{1/2}]\,\Psi) + (I \otimes N_b \Psi, H\Psi) \\
&\qquad - (I \otimes N_b^{1/2}\Psi, I \otimes H_b(I \otimes N_b^{1/2})\,\Psi) \\
&\qquad - \alpha(I \otimes N_b^{1/2}\Psi, H_I(I \otimes N_b^{1/2})\,\Psi) - \mu_0\ \|I \otimes N_b^{1/2}\Psi\|^2.
\end{aligned}$$

By (2.12), we have

$$\begin{aligned}|(I\otimes N_b^{1/2}\Psi, H_I(I\otimes N_b^{1/2})\Psi)| \leqslant C_{\theta,\varepsilon}(I\otimes N_b^{1/2}\Psi, \tilde{A}\otimes I(I\otimes N_b^{1/2})\Psi)\\ +D_{\theta,\varepsilon'}(I\otimes N_b^{1/2}\Psi, I\otimes H_b(I\otimes N_b^{1/2})\Psi)\\ +E_{\varepsilon,\varepsilon'}\|I\otimes N_b^{1/2}\Psi\|^2.\end{aligned}$$

It is not so difficult to show that, for all $f\in L^2(\mathbf{R}^\nu)$,

$$\begin{aligned}[a(f), N_b^{1/2}] &= \{(N_b+1)^{1/2}+N_b^{1/2}\}^{-1}a(f),\\ [a(f)^*, N_b^{1/2}] &= -\{(N_b-1)^{1/2}P_\Omega^\perp+N_b^{1/2}\}^{-1}a(f)^*,\end{aligned}$$

on $\mathscr{F}_0$. Hence

$$\begin{aligned}[H_I, I\otimes N_b^{1/2}]\Psi = \frac{1}{\sqrt{2}}\sum_{j=1}^J \{(B_j\otimes I)(I\otimes\{(N_b+1)^{1/2}+N_b^{1/2}\}^{-1}a(\lambda_j))\Psi\\ -(B_j\otimes I)(I\otimes\{(N_b-1)^{1/2}P_\Omega^\perp+N_b^{1/2}\}^{-1}a(\lambda_j)^*)\Psi\}.\end{aligned}$$

Using (4.4), the estimate

$$\|a(f)^*\psi\|\leqslant\|f\|_2\,\|(N_b+1)^{1/2}\psi\|,\qquad f\in L^2(\mathbf{R}^\nu),\quad \psi\in D(N_b^{1/2}),$$

and the easily proved inequalities

$$\begin{aligned}\|N_b^{1/2}\{(N_b+1)^{1/2}+N_b^{1/2}\}^{-1}\|\leqslant\tfrac{1}{2},\\ \|N_b^{1/2}\{(N_b-1)^{1/2}P_\Omega^\perp+N_b^{1/2}\}^{-1}P_\Omega^\perp\|\leqslant 1,\end{aligned}$$

we have

$$\begin{aligned}&|(I\otimes N_b^{1/2}\Psi, [H_I, I\otimes N_b^{1/2}]\Psi)|\\ &\quad\leqslant\frac{1}{\sqrt{2}}\sum_{j=1}^J\|B_j\otimes I\Psi\|\,(\|I\otimes N_b^{1/2}\{(N_b+1)^{1/2}+N_b^{1/2}\}^{-1}a(\lambda_j)\Psi\|\\ &\qquad+\|I\otimes N_b^{1/2}\{(N_b-1)^{1/2}P_\Omega^\perp+N_b^{1/2}\}^{-1}a(\lambda_j)^*\Psi\|)\\ &\quad\leqslant\frac{1}{\sqrt{2}}\sum_{j=1}^J\|B_j\otimes I\Psi\|\left(\frac{1}{2}\|I\otimes a(\lambda_j)\Psi\|+\|I\otimes a(\lambda_j)^*\Psi\|\right)\\ &\quad\leqslant\frac{1}{\sqrt{2}}\sum_{j=1}^J\|B_j\otimes I\Psi\|\left(\frac{1}{2}\|\lambda_j\|_2\,\|I\otimes N_b^{1/2}\Psi\|\right.\\ &\qquad\left.+\|\lambda_j\|_2\,\|I\otimes(N_b+1)^{1/2}\Psi\|\right).\end{aligned}$$

Combining all the estimates derived above, we obtain for all $(\theta, \varepsilon, \varepsilon') \in \mathsf{S}$

$$\begin{aligned}
&\|(\tilde{A}^{1/2} \otimes I)(I \otimes N_b^{1/2})\,\Psi\|^2 \\
&\quad \leqslant \frac{1}{1-|\alpha|\,C_{\theta,\varepsilon}} \Bigg\{ \mathrm{Re}(I \otimes N_b \Psi, H\Psi) + (|\alpha|\,E_{\varepsilon,\varepsilon'} - \mu_0)\,\|I \otimes N_b^{1/2}\Omega\|^2 \\
&\qquad + \frac{|\alpha|}{\sqrt{2}} \left(\sum_{j=1}^{J} \|B_j \otimes I\Psi\|\,\|\lambda_j\|_2 \right) \\
&\qquad \times \left(\frac{1}{2}\,\|I \otimes N_b^{1/2}\Psi\| + \|I \otimes (N_b+1)^{1/2}\,\Psi\| \right) \Bigg\}. \qquad (4.17)
\end{aligned}$$

By (4.2) with $s = 1/2$, we have

$$\|N_b^{1/2}\Phi\| \leqslant \frac{1}{\sqrt{m}}\,\|H_b^{1/2}\Phi\|, \qquad \Phi \in D(H_b^{1/2}).$$

Hence we have

$$\begin{aligned}
\|(\tilde{A}^{1/2} \otimes I)(I \otimes N_b^{1/2})\,\Psi\| &\leqslant \frac{1}{\sqrt{m}}\,\|(\tilde{A}^{1/2} \otimes I)(I \otimes H_b^{1/2})\,\Psi\| \\
&\leqslant \frac{1}{\sqrt{2m}}\,\|\tilde{H}_0 \Psi\|,
\end{aligned}$$

which, together with Lemma 3.4, gives

$$\|(\tilde{A}^{1/2} \otimes I)(I \otimes N_b^{1/2})\,\Psi\| \leqslant \frac{1}{\sqrt{2m}}\,(c\,\|H\Psi\| + d\,\|\Psi\|)$$

with constants c and d. Similarly we have

$$|(I \otimes N_b \Psi, H\Psi)| \leqslant \frac{1}{m}\,(c\,\|H\Psi\| + d\,\|\Psi\|)\,\|H\Psi\|,$$

$$\|I \otimes N_b^{1/2}\Psi\|^2 \leqslant \frac{1}{m}\,\|\Psi\|\,(c\,\|H\Psi\| + d\,\|\Psi\|),$$

$$\|B_j \otimes I\Psi\| \leqslant c'\,\|H\psi\| + d'\,\|\Psi\|$$

with constants c' and d'. By these estimates and the fact that $\mathscr{D}_\omega$ is a core of H, (4.17) extends to all $\Psi \in D(H) = D(H_0)$. In particular, taking $\Psi = \Psi_0$ and using Lemma 4.5 and (4.8), we obtain for all $(\theta, \varepsilon, \varepsilon') \in \mathsf{S}$

$$\|(\tilde{A}^{1/2}\otimes I)(I\otimes N_b^{1/2})\,\Psi_0\|^2$$
$$\leqslant \frac{1}{1-|\alpha|\,C_{\theta,\varepsilon}}\left\{\frac{\alpha^2}{2}\,(E_0(H)+|\alpha|\,E_{\varepsilon,\varepsilon'}-\mu_0)\,M_\beta(\lambda/\omega)^2\right.$$
$$\left.+\frac{|\alpha|}{2}\,M_\beta(\lambda)\left(\frac{3}{2}\,|\alpha|\,M_\beta(\lambda/\omega)+1\right)\right\}.$$

Putting this result into (4.16), which also extends to all $\Psi\in D(H)$, we have

$$\|\tilde{A}^{1/2}\otimes a(\lambda_j)\,\Psi_0\|^2\leqslant\|\lambda_j\|_2^2\,K_\beta(\theta,\varepsilon,\varepsilon',\kappa)^2$$

where $K_\beta(\theta,\varepsilon,\varepsilon',\kappa)\geqslant 0$ is defined by

$$K_\beta(\theta,\varepsilon,\varepsilon',\kappa)^2:=\frac{\alpha^2}{2}\left(\kappa+\frac{E_0(H)-\mu_0+|\alpha|\,E_{\varepsilon,\varepsilon'}}{1-|\alpha|\,C_{\theta,\varepsilon}}\right)M_\beta(\lambda/\omega)^2$$
$$+\frac{|\alpha|}{2(1-|\alpha|\,C_{\theta,\varepsilon})}\,M_\beta(\lambda)\left(\frac{3}{2}\,|\alpha|\,M_\beta(\lambda/\omega)+1\right). \qquad (4.18)$$

By this estimate and (4.15), we obtain

$$\|H_I^{(-)}\Psi_0\|\leqslant\rho_\kappa K_\beta(\theta,\varepsilon,\varepsilon',\kappa)$$

with

$$\rho_\kappa:=\frac{1}{\sqrt{2}}\sum_{j=1}^{J}\|B_j(\tilde{A}+\kappa)^{-1/2}\|\;\|\lambda_j\|_2. \qquad (4.19)$$

Hence, by (4.12), we have

$$(\Psi_0,\,Q_r^\perp\otimes P_\Omega\Psi_0)\leqslant\frac{\alpha^2}{(\mu_{r+1}-E_0(H))^2}\,\rho_\kappa^2K_\beta(\theta,\varepsilon,\varepsilon',\kappa)^2,\qquad(\theta,\varepsilon,\varepsilon')\in\mathsf{S}. \qquad (4.20)$$

4.2. *Completion of Proof of Theorem* 1.3

We are now ready to prove Theorem 1.3. We apply the following fact.

LEMMA 4.9. *Let S_n, $n=1,2,\ldots$, and S be self-adjoint operators on a Hilbert space $\mathscr{K}$ having a common core D such that, for all $\psi\in D$, $S_n\psi\to S\psi$ as $n\to\infty$. Let ψ_n be a normalized eigenvector of S_n with eigenvalue $E_n: S_n\psi_n=E_n\psi_n$ such that $E:=\lim_{n\to\infty}E_n$ and the weak limit* w-$\lim_{n\to\infty}\psi_n=\psi\neq 0$ *exist. Then ψ is an eigenvector of S with eigenvalue E. In particular, if ψ_n is a ground state of S_n, then ψ is a ground state of S.*

Proof. By the present assumption and a general theorem [21, Theorem VIII.25(a)], S_n converges to S in the strong resolvent sense as $n \to \infty$. Hence, for all $\phi \in \mathscr{K}$ and $z \in \mathbf{C} \backslash \mathbf{R}$, we have

$$
\begin{aligned}
&|(\phi, (S_n - z)^{-1} \psi_n) - (\phi, (S - z)^{-1} \psi)| \\
&\quad \leqslant |((S_n - z^*)^{-1} \phi - (S - z^*)^{-1} \phi, \psi_n)| + |((S - z^*)^{-1} \phi, \psi_n - \psi)| \\
&\quad \leqslant \|(S_n - z^*)^{-1} \phi - (S - z^*)^{-1} \phi\| + |((S - z^*)^{-1} \phi, \psi_n - \psi)| \\
&\quad \to 0 \qquad (n \to \infty),
\end{aligned}
$$

i.e., $\lim_{n \to \infty}(\phi, (S_n - z)^{-1} \psi_n) = (\phi, (S - z)^{-1} \psi)$. By the spectral theorem, we have $(\phi, (S_n - z)^{-1} \psi_n) = (E_n - z)^{-1} (\phi, \psi_n)$. Hence we obtain $(\phi, (S - z)^{-1} \psi) = (\phi, (E - z)^{-1} \psi)$ for all $\phi \in \mathscr{K}$, which implies that $(S - z)^{-1} \psi = (E - z)^{-1} \psi$. Thus $\psi \in D(S)$ and $S\psi = E\psi$. If ψ_n is a ground state of S_n, then $(\phi, S_n \phi) \geqslant E_n \|\phi\|^2$ for all $\phi \in D$. Taking the limit $n \to \infty$ in this inequality, we obtain $(\phi, S\phi) \geqslant E \|\phi\|^2$. Since D is a core for S, the last inequality extends to all $\phi \in D(S)$, which, combined with the preceding result, implies that $E = \inf \sigma(S)$. Thus ψ is a ground state of S. ∎

Let the assumption of Theorem 1.3 be satisfied. We define for $m > 0$

$$\omega_m(k) = \omega(k) + m$$

and

$$H_m := H_0(m) + \alpha H_I$$

with

$$H_0(m) = A \otimes I + I \otimes d\Gamma(\omega_m).$$

LEMMA 4.10. *The subspace $\mathscr{D}_\omega$ given by* (2.6) *is a common core of all H_m and H. Moreover, for all $\Psi \in \mathscr{D}_\omega$, $H_m \Psi \to H\Psi$ as $m \to 0$.*

Proof. The first half follows from Proposition 1.1(i) and the fact that $\mathscr{D}_\omega$ is a common core of H_0 and $H_0(m)$. The second half follows from a direct computation. ∎

Remark 4.2. By Lemma 4.10 and a general theorem [21, Theorem VIII.25(a)], H_m converges to H as $m \to 0$ in *strong resolvent sense.*

The massive Hamiltonian H_m satisfies the assumption of Theorem 1.2. Hence there exists a ground state $\Psi_0(m)$ of H_m: $H_m \Psi_0(m) = E_0(H_m) \Psi_0(m)$. Without loss of generality, we can assume that $\|\Psi_0(m)\| = 1$.

LEMMA 4.11. *We have*

$$\lim_{m\to 0} E_0(H_m)=E_0(H). \tag{4.21}$$

Proof. We have

$$H_m = H + mI \otimes N_b.$$

Hence

$$\begin{aligned} E_0(H_m) &= (\Psi_0(m), H_m \Psi_0(m)) \\ &\geqslant E_0(H) + m(\Psi_0(m), I \otimes N_b \Psi_0(m)). \end{aligned}$$

By (4.8), we have

$$(\Psi_0(m), I \otimes N_b \Psi_0(m)) \leqslant \frac{\alpha^2}{2} M_{c(m,\kappa,\mathbf{s})}(\lambda/\omega_m), \qquad s \in \mathsf{S},$$

where $c(m,\kappa,\mathbf{s}) = (c_1(m,\kappa,\mathbf{s}), ..., c_J(m,\kappa,\mathbf{s}))$ with

$$c_j(m,\kappa,\mathbf{s}) = \|B_j(\tilde{A}+\kappa)^{-1/2}\| \sqrt{\gamma_m(\mathbf{s})+\kappa}, \qquad \mathbf{s} \in \mathsf{S}, \quad \kappa > 0,$$

and $\gamma_m(\mathbf{s})$ is the constant $\gamma(\mathbf{s})$ with ω replaced by ω_m. It is easy to see that

$$\limsup_{m\to 0} \gamma_m(\theta,\varepsilon,\varepsilon') \leqslant \frac{|\alpha|\, E_{\varepsilon,\varepsilon'}}{1-|\alpha|\, C_{\theta,\varepsilon}}.$$

Hence it follows that $m(\Psi_0(m), I \otimes N_b \Psi_0(m)) \to 0$ $(m \to 0)$. Therefore

$$\liminf_{m\to 0} E_0(H_m) \geqslant E_0(H).$$

On the other hand, the strong resolvent convergence of H_m to H as $m \to 0$ (Remark 4.2) implies that $\limsup_{m\to 0} E_0(H_m) \leqslant E_0(H)$. Thus (4.21) follows. ∎

Since $\Psi_0(m)$ is a unit vector, there exists a subsequence $\{m_j\}_{j=1}^{\infty}$ with $m_1 > m_2 > \cdots > m_j \to 0$ $(j \to \infty)$ such that $\Psi_0 := \text{w-lim}_{j\to\infty} \Psi_0(m_j)$ exists. We need only to show that $\Psi_0 \neq 0$ (then, by Lemmas 4.9 and 4.10, Ψ_0 is a ground state of H). By Lemma 4.6, (4.8) and (4.20), we have for all $\mathbf{s}, \mathbf{s}' \in \mathsf{S}$ and $\kappa, \kappa' > 0$

$$(\Psi_0(m_j), Q_r \otimes P_\Omega \Psi_0(m_j)) \geqslant 1 - \frac{\alpha^2}{2} M_{c(m_j,\kappa,\mathbf{s})}(\lambda/\omega_{m_j})^2 - \frac{\alpha^2}{(\mu_{r+1} - E_0(H_{m_j}))^2} \rho_\kappa^2 K^{(j)}(\mathbf{s}, \mathbf{s}', \kappa')^2,$$

where $K^{(j)}(\mathbf{s}, \mathbf{s}', \kappa')$ is the constant $K_\beta(\mathbf{s}, \kappa')$ with ω and β replaced by ω_{m_j} and $c(m_j, \kappa, \mathbf{s})$ respectively. Since the range of $Q_r \otimes P_\Omega$ is finite dimensional, taking the limit $j \to \infty$ gives

$$(\Psi_0, Q_r \otimes P_\Omega \Psi_0) \geqslant 1 - \frac{\alpha^2}{2} M_{c(\kappa,\mathbf{s})}(\lambda/\omega)^2 - \frac{\alpha^2}{(\mu_{r+1} - E_0(H))^2} \rho_{\kappa'}^2 K_{c(\kappa,\mathbf{s})}(\mathbf{s}', \kappa')^2,$$

where $c(\kappa, \mathbf{s}) = (c_1(\kappa, \mathbf{s}), ..., c_J(\kappa, \mathbf{s}))$ with

$$c_j(\kappa, \mathbf{s}) = \|B_j(\tilde{A} + \kappa)^{-1/2}\| \sqrt{\gamma(\mathbf{s}) + \kappa}.$$

It follows that, for all $\mathbf{s}' \in \mathsf{S}$ and $\kappa' > 0$,

$$(\Psi_0, Q_r \otimes P_\Omega \Psi_0) \geqslant 1 - \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2 - \frac{\alpha^2}{(\mu_{r+1} - E_0(H))^2} \rho_{\kappa'}^2 K_\beta(\mathbf{s}', \kappa')^2. \tag{4.22}$$

If $\dim \mathscr{H} = \infty$, then $\mu_{r+1} \to \infty$ as $r \to \infty$. Hence

$$\lim_{r \to \infty} \frac{1}{(\mu_{r+1} - E_0(H))^2} = 0. \tag{4.23}$$

Therefore, under condition (1.35), the right hand side of (4.22) is positive for all sufficiently large r. Thus $\Psi_0 \neq 0$. In the case where $\dim \mathscr{H} < \infty$, using the fact that $I \otimes P_\Omega$ is finite rank and $I \otimes P_\Omega \geqslant I - I \otimes N_b$, we have

$$(\Psi_0, I \otimes P_\Omega \Psi_0) \geqslant 1 - \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2. \tag{4.24}$$

Hence, under condition (1.35), we have $\Psi_0 \neq 0$. Thus Ψ_0 is a ground state of H.

We next prove (1.36). By (4.8), we have

$$\|I \otimes N_b^{1/2} \Psi_0(m_j)\|^2 \leqslant \frac{\alpha^2}{2} M_{\beta(m_j)}(\lambda/\omega_{m_j})^2,$$

where $\beta(m)$ is the constant β with ω replaced by ω_m. We write $\Psi_0(m_j) = \{\Psi_0(m_j)^{(n)}\}_{n=0}^{\infty}$ with $\Psi_0(m_j)^{(n)} \in L^2_{\mathrm{sym}}(\mathbf{R}^{\nu n}; \mathscr{H})$. Then, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} n \, \|\Psi_0(m_j)^{(n)}\|^2 \leqslant \frac{\alpha^2}{2} M_{\beta(m_j)}(\lambda/\omega_{m_j})^2.$$

Let $\{\Psi_l^{(n)}\}_{l=1}^{\infty}$ be a complete orthonormal system of $L^2_{\mathrm{sym}}(\mathbf{R}^{\nu n}; \mathscr{H})$. Then, for all $M \geqslant 1$,

$$\sum_{n=1}^{N} \sum_{l=1}^{M} n \, |(\Psi_0(m_j)^{(n)}, \Psi_l^{(n)})|^2 \leqslant \frac{\alpha^2}{2} M_{\beta(m_j)}(\lambda/\omega_{m_j})^2. \tag{4.25}$$

It is easy to see that $\lim_{j\to\infty}(\Psi_0(m_j)^{(n)}, \Psi_l^{(n)}) = (\Psi_0^{(n)}, \Psi_l^{(n)})$ and

$$\limsup_{j\to\infty} M_{\beta(m_j)}(\lambda/\omega_{m_j}) \leqslant M_\beta(\lambda/\omega).$$

Hence, taking the limit $j \to \infty$ in (4.25), we have

$$\sum_{n=1}^{N} \sum_{l=1}^{M} n \, |(\Psi_0^{(n)}, \Psi_l^{(n)})|^2 \leqslant \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2.$$

In this inequality, taking the limit $M \to \infty$ first and then the limit $N \to \infty$, we obtain

$$\sum_{n=1}^{\infty} n \, \|\Psi_0^{(n)}\|^2 \leqslant \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2,$$

which implies that $\Psi_0 \in D(I \otimes N_b^{1/2})$ and

$$\|I \otimes N_b^{1/2} \Psi_0\|^2 \leqslant \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2.$$

Since $Q_r \otimes P_\Omega \leqslant I$ and $I \otimes P_\Omega \leqslant I$, it follows from (4.22)–(4.24) that

$$\|\Psi_0\|^2 \geqslant 1 - \frac{\alpha^2}{2} M_\beta(\lambda/\omega)^2.$$

Thus (1.36) is obtained.

5. PROOF OF PROPOSITION 1.4

Suppose that the assumption of Proposition 1.4 is satisfied. It is well known that, for all $f \in L^2(\mathbf{R}^\nu)$, the operator

$$P(f) := \frac{i}{\sqrt{2}} \{a(f)^* - a(f)\}$$

is essentially self-adjoint on the finite particle subspace $\mathscr{F}_0$ of $\mathscr{F}_b$. It is easy to see that

$$T := \sum_{j=1}^{J} B_j \otimes P(\lambda_j/\omega)$$

is essentially self-adjoint on $\mathscr{H} \hat{\otimes} \mathscr{F}_0$. We denote the closure of T by the same symbol. Then we have a unitary operator

$$U := e^{i\alpha T}$$

The following lemma can be easily proven:

Lemma 5.1.

$$U(I \otimes H_b)\, U^{-1} = I \otimes H_b + \alpha H_I + \alpha^2 R_B \otimes I,$$

where R_B *is defined by* (1.37).

We introduce a new Hamiltonian

$$\hat{H} := I \otimes H_b + U^{-1}(A \otimes I)\, U - \alpha^2 R_B \otimes I. \tag{5.1}$$

Then, by Lemma 5.1, we have the operator equality

$$H(\alpha) = U\hat{H}U^{-1}. \tag{5.2}$$

In particular,

$$E_0(H(\alpha)) = E_0(\hat{H}). \tag{5.3}$$

We now proceed to proof of Proposition 1.1. Since (5.3) holds, we estimate $E_0(\hat{H})$. By (5.1), we have

$$\hat{H} \geqslant \mu_0 - \alpha^2 \|R_B\|,$$

which, together with (5.3), implies the left hand side inequality of (1.41).

By the variational principle, we have for all $u \in \mathscr{H}$ with $\|u\| = 1$

$$E_0(\hat{H}) \leqslant (u \otimes \Omega, \hat{H} u \otimes \Omega) = (Uu \otimes \Omega, (A \otimes I)\, Uu \otimes \Omega) - \alpha^2 (u, R_B u).$$

Since $u \otimes \Omega$ is an entire analytic vector of T, we have

$$\begin{aligned}(Uu \otimes \Omega, (A \otimes I)\, Uu \otimes \Omega) &= \sum_{m,n=0}^{\infty} \sum_{1 \leqslant j_1, ..., j_m, i_1, ..., i_n \leqslant J} \frac{1}{m!} \frac{1}{n!} ((i\alpha)^m B_{j_1} \cdots B_{j_m} u, A (i\alpha)^n B_{i_1} \cdots B_{i_n} u) \\ &\quad \times (P(\lambda_{j_1}/\omega) \cdots P(\lambda_{j_m}/\omega)\Omega, P(\lambda_{i_1}/\omega) \cdots P(\lambda_{i_n}/\omega)\Omega).\end{aligned}$$

We denote the closure of $P(f)$ by the same symbol. Since Λ_{ij}, $i, j = 1, ..., J$, are real by (A.7), we have

$$e^{itP(\lambda_i/\omega)} e^{isP(\lambda_j/\omega)} = e^{isP(\lambda_j/\omega)} e^{itP(\lambda_i/\omega)}, \qquad i, j = 1, ..., J,$$

for all $s, t \in \mathbf{R}$ (see [22, Theorem X.41]). It follows that the self-adjoint operators $P(\lambda_j/\omega)$, $j = 1, ..., J$, are strongly commuting[2] and

$$e^{i \sum_{j=1}^{J} t_j P(\lambda_j/\omega)} = t^{iP(\sum_{j=1}^{J} t_j \lambda_j/\omega)} \tag{5.4}$$

for all $t_j \in \mathbf{R}$, $j = 1, ..., J$. Hence there exists a unique J-dimensional spectral measure $E(dp)$ such that

$$E(S_1 \times \cdots \times S_J) = E_1(S_1) \cdots E_J(S_J)$$

for all Borel sets $S_j \subset \mathbf{R}$, $j = 1, ..., J$, where E_j is the spectral measure of $P(\lambda_j/\omega)$. We have

$$\begin{aligned}&(P(\lambda_{j_i}/\omega) \cdots P(\lambda_{j_m})/\omega)\Omega, P(\lambda_{i_1}/\omega) \cdots P(\lambda_{i_n})/\omega)\Omega) \\ &\quad = \int_{\mathbf{R}^J} p_{j_1} \cdots p_{j_m} p_{i_1} \cdots p_{i_n} (\Omega, E(dp)\Omega).\end{aligned}$$

By the well known formula

$$(\Omega, e^{iP(f)}\Omega) = e^{-\|f\|_2^2/4}, \qquad f \in L^2(\mathbf{R}^\nu)),$$

and (5.4), we have

$$(\Omega, e^{i \sum_{j=1}^{J} t_j P(\lambda_j/\omega)}\Omega) = e^{-\sum_{i,j=1}^{J} \Lambda_{ij} t_i t_j/4}, \qquad t_i \in \mathbf{R}, \quad i = 1, ..., J.$$

[2] Two self-adjoint operators S and T on a Hilbert space are said to *strongly commute* if their spectral measures commute. A characterization of strong commutativity is given in [21, Theorem VIII.13].

The left hand side is equal to $\int_{\mathbf{R}^J} e^{i\sum_{j=1}^J t_j p_j}(\Omega, E(dp)\Omega)$. Hence we obtain

$$(\Omega, E(dp)\Omega) = \frac{1}{\pi^{J/2}\sqrt{\det \Lambda}}\, e^{-(p, \Lambda^{-1}p)}\, dp, \qquad p \in \mathbf{R}^J.$$

This implies that, for all $t > 0$,

$$\int_{\mathbf{R}^J} e^{t\sum_{i=1}^J |p_i|\, \|B_i\|}(\Omega, E(dp)\Omega) < \infty.$$

It follows that

$$\begin{aligned}&(Uu \otimes \Omega, (A \otimes I)\, Uu \otimes \Omega)\\ &\quad = \frac{1}{\pi^{J/2}\sqrt{\det \Lambda}} \int_{\mathbf{R}^J} (u, e^{-i\alpha B(p)} A e^{i\alpha B(p)} u)\, e^{-(p, \Lambda^{-1}p)}\, dp.\end{aligned}$$

Thus we obtain

$$E_0(\hat{H}) \leqslant (u, L(\alpha)u),$$

which, together with the variational principle and (5.3), implies the right hand side inequality of (1.41).

By (1.41) and (1.43), we have

$$\frac{\mu_0}{\alpha^2} \leqslant \frac{E_0(H(\alpha))}{\alpha^2} + \|R_B\| \leqslant \frac{\|A\|}{\alpha^2}.$$

Then, taking the limit $|\alpha| \to \infty$, we obtain (1.42).

Remark 5.1. Let

$$\hat{H}(\kappa) = \kappa I \otimes H_b + U^{-1}(A \otimes I)\, U - \alpha^2 R_B \otimes I, \qquad \kappa > 0.$$

Then, applying [6, Theorem 2.12], we can show that, for all $z \in \mathbf{C} \backslash \mathbf{R}$,

$$\operatorname*{s-lim}_{\kappa \to \infty} (\hat{H}(\kappa) - z)^{-1} = (L(\alpha) - z)^{-1} \otimes P_\Omega,$$

where s-lim denotes strong limit. This gives a connection of $L(\alpha)$ with $\hat{H}$, i.e., $L(\alpha)$ is obtained as a scaling limit of $\hat{H}$.

6. PROOF OF THEOREM 1.5

The basic idea of proof is to use the mini-max principle for $\hat{H}$ defined by (5.1). Suppose that the assumption of Theorem 1.5 is satisfied. Let

$$\mu_2(\hat{H}) := \sup_{\Phi \in \mathscr{F}} U_{\hat{H}}(\Phi)$$

with

$$U_{\hat{H}} = \inf_{\Psi \in D(\hat{H}),\ \|\Psi\|=1,\ \Psi \in [\Phi]^{\perp}} (\Psi, \hat{H}\Psi),$$

where $[\Phi]^{\perp} = \{\Psi \mid (\Psi, \Phi) = 0\}$. We estimate $\mu_2(\hat{H})$ from below. For this purpose, we write

$$\hat{H} = I \otimes H_b + A \otimes I + W - \alpha^2 R_B \otimes I$$

where

$$W := U^{-1}(A \otimes I)\, U - A \otimes I.$$

LEMMA 6.1. *For all $\varepsilon > 0$ and $\Psi \in D(I \otimes H_b)$,*

$$|(\Psi, W\Psi)| \leqslant \varepsilon(\Psi, (I \otimes H_b)\,\Psi) + C_\varepsilon(\alpha)\, \|\Psi\|^2, \tag{6.1}$$

where $C_\varepsilon(\alpha)$ is defined by (1.47).

Proof. We write as

$$W = (U^{-1} - I)(A \otimes I)\, U + (A \otimes I)(U - I).$$

For all $\Psi \in \bigcap_{j=1}^{J} D(B_j \otimes P(\lambda_j/\omega)) \subset D(T)$, we have by the functional calculus (cf. [7, 8, Lemma 4.3])

$$\|(U^{\pm 1} - I)\,\Psi\| \leqslant |\alpha| \sum_{j=1}^{J} \|B_j \otimes P(\lambda_j/\omega)\,\Psi\|.$$

Let $\Psi \in D(I \otimes H_b)$ with $\|\Psi\| = 1$. Then $\Psi \in D(B_j \otimes P(\lambda_j/\omega))$, $j = 1, ..., J$. By (A.7) and [22, Theorem X.41], one can show that, for all $s, t \in \mathbf{R}$

$$e^{isB_i \otimes P(\lambda_i/\omega)} e^{itB_j \otimes P(\lambda_j/\omega)} = e^{itB_j \otimes P(\lambda_j/\omega)} e^{isB_i \otimes P(\lambda_i/\omega)}, \qquad i, j = 1, ..., J.$$

Hence, by [21, Theorem VIII.13], $B_j \otimes P(\lambda_j/\omega)$, $j = 1, ..., J$ are strongly commuting self-adjoints operators. In particular, $UB_j \otimes P(\lambda_j/\omega) \subset B_j \otimes P(\lambda_j/\omega)\, U$ and $U\Psi \in D(B_j \otimes P(\lambda_j/\omega))$, $j = 1, ..., J$. Using these facts, we can estimate $|(\Psi, W\Psi)|$ as follows:

$$
\begin{aligned}
|(\Psi, W\Psi)| &\leqslant \| W\Psi\| \\
&\leqslant \sum_{j=1}^{J} |\alpha| \, \{ \|(B_j \otimes P(\lambda_j/\omega))(A \otimes I) \, U\Psi\| \\
&\quad + \|A\| \, \|B_j \otimes P(\lambda_j/\omega) \, \Psi\| \} \\
&= \sum_{j=1}^{J} |\alpha| \, \{ \|(B_j A \otimes I)(I \otimes P(\lambda_j/\omega)) \, U\Psi\| \\
&\quad + \|A\| \, \|(B_j \otimes I)(I \otimes P(\lambda_j/\omega)) \, \Psi\| \} \\
&\leqslant \sum_{j=1}^{J} |\alpha| \, (\|B_j A\| + \|A\| \, \|B_j\|) \, \|(I \otimes P(\lambda_j/\omega)) \, \Psi\| \\
&\leqslant \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} (\|B_j A\| + \|A\| \, \|B_j\|) \\
&\quad \times (\|I \otimes a(\lambda_j/\omega)^* \, \Psi\| + \|I \otimes a(\lambda_j/\omega) \, \Psi\|) \\
&\leqslant \frac{|\alpha|}{\sqrt{2}} \sum_{j=1}^{J} (\|B_j A\| + \|A\| \, \|B_j\|) \\
&\quad \times (2 \, \|\lambda_j/\omega^{3/2}\|_2 \, \|I \otimes H_b^{1/2} \Psi\| + \|\lambda_j/\omega\|_2 \, \|\Psi\|) \\
&\leqslant \varepsilon(\Psi, I \otimes H_b \Psi) + C_\varepsilon(\alpha).
\end{aligned}
$$

Thus (6.1) follows. ∎

LEMMA 6.2. *Let $M(\alpha)$ be given by* (1.50). *Then*

$$\mu_2(\hat{H}) \geqslant M(\alpha).$$

Proof. Let e_0 be a normalized ground state e_0 of A: $Ae_0 = \mu_0 e_0$, $\|e_0\| = 1$, and set

$$\Phi_0 \equiv e_0 \otimes \Omega.$$

We denote by $\{e_n\}_{n=1}^{N}$ $(N \leqslant \infty)$ a complete orthonormal system of $[e_0]^\perp$. Then we have

$$
\begin{aligned}
\mathscr{F} &= \bigoplus_{n=0}^{N} \mathscr{L}\{e_n\} \otimes \mathscr{F}_b \\
&= \left\{ \Psi = \{e_n \otimes \Theta_{[n]}\}_{n=0}^{N} \mid \Theta_{[n]} \in \mathscr{F}_b, \, 0 \leqslant n \leqslant N, \, \sum_{n=0}^{N} \|\Theta_{[n]}\|^2 < \infty \right\}.
\end{aligned}
$$

We denote $\Theta_{[n]} \in \mathscr{F}_b$ as $\Theta_{[n]} = \{\Theta_{[n]}^{(0)}, \Theta_{[n]}^{(1)}, \Theta_{[n]}^{(2)}, ...\}$ with $\Theta_{[n]}^{(j)} \in \bigotimes_s^j L^2(\mathbf{R}^\nu)$. We have

$$[\Phi_0]^\perp = \{\Psi = \{e_n \otimes \Theta_{[n]}\}_{n=0}^N \in \mathscr{F} \mid \Theta_{[n]} \in \mathscr{F}_b \text{ with } \Theta_{[0]}^{(0)} = 0\}.$$

For all $\Psi = \{e_n \otimes \Theta_{[n]}\}_{n=0}^N \in [\Phi_0]^\perp$ with $\|\Psi\|^2 = \sum_{n=0}^N \|\Theta_{[n]}\|^2 = 1$ and $\Psi \in D(I \otimes H_b)$ (hence $\Theta_{[n]} \in D(H_b)$ with $\sum_{n=0}^N \|H_b \Theta_{[n]}\|^2 < \infty$), we have

$$\begin{aligned}(\Psi, (I \otimes H_b)\Psi) &= \sum_{n=0}^N (\Theta_{[n]}, H_b \Theta_{[n]}) \\ &= \sum_{n=0}^N \sum_{j=1}^\infty (\Theta_{[n]}^{(j)}, H_b \Theta_{[n]}^{(j)}) \\ &\geqslant m \sum_{n=0}^N \sum_{j=1}^\infty \|\Theta_{[n]}^{(j)}\|^2 \\ &\geqslant m \sum_{j=1}^\infty \|\Theta_{[0]}^{(j)}\|^2 \\ &= m \|\Theta_{[0]}\|^2,\end{aligned}$$

where we have used the condition $\Theta_{[0]}^{(0)} = 0$. Further, it is easy to see that

$$(\Psi, (A \otimes I)\Psi) \geqslant \mu_0 \|\Theta_{[0]}\|^2 + \mu_1 \sum_{n=1}^N \|\Theta_{[n]}\|^2.$$

Therefore, by Lemma 6.1, we have

$$\begin{aligned}(\Psi, \hat{H}\Psi) \geqslant{}& m(1-\varepsilon) \|\Theta_{[0]}\|^2 + \mu_0 \|\Theta_{[0]}\|^2 \\ &+ \mu_1 \sum_{n=1}^N \|\Theta_{[n]}\|^2 - C_\varepsilon(\alpha) - \alpha^2 (\Psi, R_B \otimes I\Psi) \\ \geqslant{}& K_\varepsilon(\alpha) - \alpha^2 \|R_B\|.\end{aligned}$$

which implies that for all $\varepsilon > 0$

$$\mu_2(\hat{H}) \geqslant K_\varepsilon(\alpha) - \alpha^2 \|R_B\|.$$

Therefore, taking the supremum w.r.t. $\varepsilon > 0$, we obtain the desired result. ∎

We are now ready to prove Theorem 1.5. By (1.52) and Lemma 6.2, we have $E_0(\hat{H}) < \mu_2(\hat{H})$. This estimate and the min-max principle [23, Theorem XIII.1] imply that $E_0(\hat{H})$ is the first eigenvalue of $\hat{H}$. Hence $\hat{H}$ has a ground state. The min-max principle also tells us the following: if

$\mu_2 := \mu_2(\hat{H})$ is the bottom of the essential spectrum of $\hat{H}$, then there is at most one eigenvalue below μ_2. If μ_2 is not the bottom of the essential spectrum of $\hat{H}$, then μ_2 is the second eigenvalue of $\hat{H}$. Hence, in either case, $E_0(\hat{H})$ is a simple eigenvalue of $\hat{H}$. Thus the ground state of $\hat{H}$ is unique. This completes the proof of Theorem 1.5.

7. PROOF OF THEOREM 1.6

Suppose that the assumption of Theorem 1.6 is satisfied. Let

$$\mu_{N+1}(\hat{H}) := \sup_{\Phi_1, ..., \Phi_N \in \mathscr{F}} U_{\hat{H}}(\Phi_1, ..., \Phi_N)$$

with

$$U_{\hat{H}}(\Phi_1, ..., \Phi_N) = \inf_{\Psi \in D(\hat{H});\ \|\Psi\| = 1,\ \Psi \in [\Phi_1, ..., \Phi_N]^\perp} (\Psi, \hat{H}\Psi),$$

where $[\Phi_1, ..., \Phi_N]^\perp$ denotes the orthogonal complement of $\mathscr{L}\{\Phi_n \mid n=1, ..., N\}$. Let $\{w_n\}_{n=1}^N$ be a complete orthonormal basis of $\mathscr{H}$ and set

$$\Phi_n = w_n \otimes \Omega, \qquad n = 1, ..., N.$$

For $\Psi \in [\Phi_1, ..., \Phi_N]^\perp \cap D(I \otimes H_b)$ with $\|\Psi\| = 1$, we have $\Psi \in \bigcap_{n=1}^N [\Phi_n]^\perp$. We write Ψ as

$$\Psi = \sum_{n=1}^N w_n \otimes \Theta_{[n]},$$

with $\Theta_{[n]} \in \mathscr{F}_b$. It is easy to see that

$$[\Phi_1, ..., \Phi_N]^\perp = \left\{ \Psi = \sum_{n=1}^N w_n \otimes \Theta_{[n]} \in \mathscr{F} \;\middle|\; \Theta_{[n]} \in \mathscr{F}_b \text{ with } \Theta_{[n]}^{(0)} = 0,\ n = 1, ..., N \right\}.$$

In the same way as in the proof of Lemma 6.2, we have

$$(\Psi, (I \otimes H_b)\Psi) \geqslant m \sum_{n=1}^N \sum_{j=1}^\infty \|\Theta_{[n]}^{(j)}\|^2 = m\,\|\Psi\|^2.$$

Since $\mu_0 \leqslant U^{-1}(A \otimes I)\,U$, we have

$$(\Psi, \hat{H}\Psi) \geqslant m + \mu_0 - \alpha^2\,\|R_B\|.$$

Hence we obtain

$$\mu_{N+1}(\hat{H}) \geqslant m + \mu_0 - \alpha^2 \|R_B\|.$$

By (1.41) and (1.50), we have

$$E_0(\hat{H}) < \mu_{N+1}(\hat{H}).$$

Hence, by the min-max principle, we have the following alternative: (i) if $\mu_{N+1} := \mu_{N+1}(\hat{H})$ is the bottom of the essential spectrum of $\hat{H}$, then there are at most N eigenvalues (counting multiplicity) below μ_{N+1}; (ii) if $\mu_{N+1}(\hat{H})$ is not the bottom of the essential spectrum of $\hat{H}$, then μ_{N+1} is the $(N+1)$th eigenvalue of $\hat{H}$ below the bottom of the essential spectrum of $\hat{H}$. In either case, there are at most N eigenvalues below μ_{N+1}. Thus the desired result follows.

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REFERENCES

1. A. Amann, Ground states of a spin-boson model, *Ann. Phys.* **208** (1991), 414–448.
2. A. Arai, On a model of a harmonic oscillator coupled to a quantized, massless, scalar field, I, *J. Math. Phys.* **22** (1981), 2539–2548.
3. A. Arai, Rigorous theory of spectra and radiation for a model in quantum electrodynamics, *J. Math. Phys.* **24** (1981), 1896–1910.
4. A. Arai, Spectral analysis of a quantum harmonic oscillator coupled to infinitely many scalar bosons, *J. Math. Anal. Appl.* **140** (1989), 270–288.
5. A. Arai, Perturbation of embedded eigenvalues: A general class of exactly soluble models in Fock spaces, *Hokkaido Math. J.* **19** (1990), 1–34.
6. A. Arai, An asymptotic analysis and its application to the non-relativistic limit of the Pauli–Fierz and a spin-boson model, *J. Math. Phys.* **31** (1990), 2653–2663.
7. A. Arai and M. Hirokawa, "On the Existence and Uniqueness of Ground States of the Spin-Boson Hamiltonian", Hokkaido University Preprint Series in Mathematics, No. 309, 1995.
8. A. Arai and M. Hirokawa, On the spin-boson model, *in* "Proceedings of the Symposium Quantum Stochastic Analysis and Related Fields Held at the RIMS, Kyoto University, November 27–29, 1995," RIMS Kokyuroku, No. 957, pp. 16–35, 1996.
9. V. Bach, J. Fröhlich, and I. Sigal, Mathematical theory of nonrelativistic matter and radiation, *Lett. Math. Phys.* **34** (1995), 183–201.
10. V. Bach, J. Fröhlich, and I. Sigal, Quantum electrodynamics of confined nonrelativistic particles, preprint, 1996.

11. E. B. Davies, Symmetry breaking for molecular open system, *Ann. Inst. H. Poincaré A* **35** (1981), 149–171.
12. M. Fannes, B. Nachtergaele, and A. Verbeure, The equilibrium state of the spin-boson model, *Comm. Math. Phys.* **114** (1988), 537–548.
13. J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschr. Phys.* **22** (1973), 159–198.
14. J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. H. Poincaré A* **19** (1973), 1–103.
15. C. Gérard, Asymptotic completeness for the spin-boson model with a particle number cutoff, *Rev. Math. Phys.* **8** (1996), 549–589.
16. J. Glimm and A. Jaffe, The $\lambda(\varphi^4)_2$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum, *Ann. of Math.* **91** (1970), 362–401.
17. M. Hübner and H. Spohn, Spectral properties of the spin-boson Hamiltonian, *Ann. Inst. H. Poincaré* **62** (1995), 289–323.
18. M. Hübner and H. Spohn, Radiative decay: Nonperturbative approach, *Rev. Math. Phys.* **7** (1995), 363–387.
19. V. Jakšić and C. A. Pillet, On a model for quantum friction. I. Fermi's golden rule and dynamics at zero temperature, *Ann. Inst. H. Poincaré* **62** (1995), 47–68.
20. V. Jakšić and C. A. Pillet, On a model for quantum friction. II. Fermi's golden rule and dynamics at positive temperature, *Comm. Math. Phys.* **176** (1996), 619–644.
21. M. Reed and B. Simon, "Methods of Modern Mathematical Physics," Vol. I, Academic Press, New York, 1972.
22. M. Reed and B. Simon, "Methods of Modern Mathematical Physics," Vol. II, Academic Press, New York, 1974.
23. M. Reed and B. Simon, "Methods of Modern Mathematical Physics," Vol. IV, Academic Press, New York, 1978.
24. B. Simon and R. Hoegh-Krohn, Hypercontractive semigroups and two dimensional self-coupled Bose fields, *J. Funct. Anal.* **9** (1972), 121–180.
25. H. Spohn, Ground state(s) of the spin-boson Hamiltonian, *Comm. Math. Phys.* **123** (1989), 277–304.
26. H. Spohn, Asymptotic completeness for Rayleigh scattering, preprint, 1996.